Consistent explicit staggered schemes for compressible flows Part I: the barotropic Euler equations.

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CONSISTENT EXPLICIT STAGGERED SCHEMES FOR COMPRESSIBLE FLOWS

PART I: THE BAROTROPIC EULER EQUATIONS.

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Abstract. In this paper, we build and analyze the stability and consistency of an explicit scheme for the compressible barotropic Euler equations. This scheme is based on a staggered space discretization, with an upwinding performed with respect to the material velocity only (so that, in particular, the pressure gradient term is centered). The velocity convection term is built in such a way that the solutions satisfy a discrete kinetic energy balance, with a remainder term at the left-hand side which is shown to be non-negative under a CFL condition. Then, in one space dimension, we prove that if the solutions to the scheme converge to some limit as the time and space step tend to zero, then this limit is an entropy weak solution of the continuous problem. Numerical tests confirm this theory, and show in addition (in 1D, and thus in absence of contact discontinuities) a first-order convergence rate.

Key words. Finite volumes, finite elements, staggered discretizations, barotropic Euler equations, shallow-water equations, compressible flows, analysis.

AMS subject classifications. 65M12

1. Introduction. We address in this work the numerical solution of the so-called barotropic Euler equations, which read:

$$\partial_t \rho + \operatorname{div}(\rho \boldsymbol{u}) = 0, \tag{1.1a}$$

$$\partial_t(\rho \boldsymbol{u}) + \operatorname{div}(\rho \boldsymbol{u} \otimes \boldsymbol{u}) + \boldsymbol{\nabla} p = 0, \tag{1.1b}$$

$$p = \wp(\rho) = \rho^\gamma, \tag{1.1c}$$

where t stands for the time, ρ, $\boldsymbol{u}$ and p are the density, velocity and pressure in the flow, and $\gamma \geq 1$ is a coefficient specific to the considered fluid. The problem is supposed to be posed over $\Omega \times (0, T)$, where Ω is an open bounded connected subset of $\mathbb{R}^d$, $1 \leq d \leq 3$, and $(0, T)$ is a finite time interval. This system must be supplemented by initial conditions for ρ and $\boldsymbol{u}$, denoted by ρ_0 and $\boldsymbol{u}_0$, and we assume $\rho_0 > 0$. It must also be supplemented by a suitable boundary condition, which we suppose to be:

$$\boldsymbol{u} \cdot \boldsymbol{n} = 0, \text{ at any time and } a.e. \text{ on } \partial\Omega,$$

where $\boldsymbol{n}$ stands for the normal vector to the boundary.

Let us denote by E_k the kinetic energy $E_k = \frac{1}{2}\,|\boldsymbol{u}|^2$. Taking the inner product of (1.1b) by $\boldsymbol{u}$ yields, after formal compositions of partial derivatives and using the mass balance (1.1a):

$$\partial_t(\rho E_k) + \operatorname{div}\big(\rho\, E_k\, \boldsymbol{u}\big) + \boldsymbol{\nabla} p \cdot \boldsymbol{u} = 0. \tag{1.2}$$

This relation is referred to as the kinetic energy balance.

Let us now define the function $\mathcal{P}$, from $(0, +\infty)$ to $\mathbb{R}$, as a primitive of $s \mapsto \wp(s)/s^2$; this quantity is often called the elastic potential. Let $\mathcal{H}$ be the function defined by $\mathcal{H}(s) = s\mathcal{P}(s)$, $\forall s \in (0, +\infty)$. For the specific equation of state $\wp$ used here, we obtain:

$$\mathcal{H}(s) = s\,\mathcal{P}(s) = \begin{cases} \dfrac{s^\gamma}{\gamma - 1} \text{ if } \gamma > 1, \\ s\ln(s) \text{ if } \gamma = 1. \end{cases} . \tag{1.3}$$

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Since $\wp$ is an increasing function, $\mathcal{H}$ is convex. In addition, it may easily be checked that $\rho\mathcal{H}'(\rho) - \mathcal{H}(\rho) = \wp(\rho)$. Therefore, by a formal computation, detailed in the appendix, multiplying (1.1a) by $\mathcal{H}'(\rho)$ yields:

$$\partial_t\big(\mathcal{H}(\rho)\big) + \mathrm{div}\big(\mathcal{H}(\rho)\,\boldsymbol{u}\big) + p\,\mathrm{div}(\boldsymbol{u}) = 0. \tag{1.4}$$

Let us denote by $\mathcal{S}$ the quantity $\mathcal{S} = \rho E_k + \mathcal{H}(\rho)$. Summing (1.2) and (1.4), we get:

$$\partial_t\mathcal{S} + \mathrm{div}\big((\mathcal{S}+p)\,\boldsymbol{u}\big) = 0. \tag{1.5}$$

In fact, to avoid invoking unrealistic regularity assumption, such a computation should be done on regularized equations (obtained by adding diffusion perturbation terms, see *e.g.* [7, Introduction, Section 3.2]), and, when making these regularization terms tend to zero, positive measures appear at the left-hand-side of (1.5), so that we get in the distribution sense:

$$\partial_t\mathcal{S} + \mathrm{div}\big((\mathcal{S}+p)\,\boldsymbol{u}\big) \leq 0. \tag{1.6}$$

The quantity $\mathcal{S}$ is an entropy of the system, and an entropy solution to (1.1) is thus required to satisfy:

$$\int_0^T \int_\Omega \big[-\mathcal{S}\partial_t\varphi - (\mathcal{S}+p)\,\boldsymbol{u}\cdot\boldsymbol{\nabla}\varphi\big]\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t - \int_\Omega \mathcal{S}_0\;\varphi(\boldsymbol{x},0)\,\mathrm{d}\boldsymbol{x} \leq 0,$$
$$\forall\varphi \in \mathrm{C}_c^\infty\big(\Omega\times[0,T)\big),\;\; \varphi \geq 0, \tag{1.7}$$

with $\mathcal{S}_0 = \frac{1}{2}\rho_0|\boldsymbol{u}_0|^2 + \mathcal{H}(\rho_0)$. Then, since the normal velocity is prescribed to zero at the boundary, integrating (1.6) over Ω yields:

$$\frac{d}{dt}\int_\Omega \big[\frac{1}{2}\,\rho\,|\boldsymbol{u}|^2 + \mathcal{H}(\rho)\big]\,\mathrm{d}\boldsymbol{x} \leq 0. \tag{1.8}$$

Since $\rho \geq 0$ by (1.1a) (and the associated initial and boundary conditions) and the function $s \mapsto \mathcal{H}(s)$ is bounded by below and increasing at least for s large enough, Inequality (1.8) provides an estimate on the solution.

The purpose of this paper is to build an explicit scheme for the numerical solution of System (1.1). This scheme is, in fact, an explicit variant of a recent all-Mach-number pressure correction scheme [5, 12] implemented in the open-source software ISIS [16], and is developed with the aim to offer an efficient alternative for quickly varying unstationary flows, with a characteristic Mach number in the range or greater than the unity. The proposed algorithm thus keeps the space discretizations used in [5, 12], namely staggered finite volume or finite element discretizations. This discretization precludes the use of Riemann solvers (see *e.g.* [20, 7, 2] for textbooks on this latter technique), and we thus implement the most naive upwinding, with respect to the material velocity only (similarly to what is proposed in the collocated context in the AUSM method [18, 17], although with a simpler upwinding algorithm). The pressure gradient is defined as the transpose of the natural velocity divergence, and is thus centered. Last but not least, the velocity convection term is built in such a way to allow to derive a discrete kinetic energy balance.

We prove the following results for this scheme:

- a discrete kinetic energy balance (*i.e.* a discrete analogue of (1.2)) is established on dual cells, while a discrete potential elastic balance (*i.e.* a discrete analogue of (1.4)) is established on primal cells.
  Note however that, because of residual terms appearing in the potential elastic balance, contrary to what is obtained for implicit and semi-implicit variants of the present scheme [5, 12], these equations do not seem to yield the stability of the scheme (*i.e.* a discrete global entropy conservation analogue to Equation (1.8)), at least unless supposing drastic limitations of the time step.

- Second, in one space dimension, the limit of any convergent sequence of solutions to the scheme is shown to be a weak solution to the continuous problem, and thus to satisfy the Rankine-Hugoniot conditions.
- Finally, still in one space dimension, passing to the limit in the discrete kinetic energy and elastic potential balances, such a limit is also shown to satisfy the entropy inequality (1.7).

This paper is structured as follows. We begin with the presentation of the space discretization (Section 2), then the scheme is given (Section 3), and we derive the discrete kinetic and elastic potential balances satisfied by its solutions (Section 4). The next section is dedicated to the proof, in 1D, of the consistency of the scheme (Section 5). We then present some numerical tests, to assess the behaviour of the algorithm (Section 6). The discrete kinetic energy and elastic potential balances are obtained as particular cases of more general results concerning the explicit finite volume discretization of transport operators, which are established in the appendix.

The results presented in this work are extended in a companion paper [15] to the "full" Euler equations.

2. Meshes and unknowns. In this section, we focus on the discretization of a multi-dimensional domain (*i.e.* $d = 2$ or $d = 3$); the extension to the one-dimensional case is straightforward (see Section 5).

Let $\mathcal{M}$ be a decomposition of the domain Ω, supposed to be regular in the usual sense of the finite element literature (*e.g.* [3]). The cells may be:

- for a general domain Ω, either non-degenerate quadrilaterals ($d = 2$) or hexahedra ($d = 3$) or simplices, both type of cells being possibly combined in a same mesh,
- for a domain the boundaries of which are hyperplanes normal to a coordinate axis, rectangles ($d = 2$) or rectangular parallelepipeds ($d = 3$) (the faces of which, of course, are then also necessarily normal to a coordinate axis).

By $\mathcal{E}$ and $\mathcal{E}(K)$ we denote the set of all $(d-1)$-faces σ of the mesh and of the element $K \in \mathcal{M}$ respectively. The set of faces included in the boundary of Ω is denoted by $\mathcal{E}_{\rm ext}$ and the set of internal faces (*i.e.* $\mathcal{E} \setminus \mathcal{E}_{\rm ext}$) is denoted by $\mathcal{E}_{\rm int}$; a face $\sigma \in \mathcal{E}_{\rm int}$ separating the cells K and L is denoted by $\sigma = K|L$. The outward normal vector to a face σ of K is denoted by $\boldsymbol{n}_{K,\sigma}$. For $K \in \mathcal{M}$ and $\sigma \in \mathcal{E}$, we denote by $|K|$ the measure of K and by $|\sigma|$ the $(d-1)$-measure of the face σ. For $1 \leq i \leq d$, we denote by $\mathcal{E}^{(i)} \subset \mathcal{E}$ and $\mathcal{E}_{\rm ext}^{(i)} \subset \mathcal{E}_{\rm ext}$ the subset of the faces of $\mathcal{E}$ and $\mathcal{E}_{\rm ext}$, respectively, which are perpendicular to the i^{th} unit vector of the canonical basis of $\mathbb{R}^d$.

The space discretization is staggered, using either the Marker-And Cell (MAC) scheme [11, 10], or nonconforming low-order finite element approximations, namely the Rannacher and Turek element (RT) [19] for quadrilateral or hexahedric meshes, or the lowest degree Crouzeix-Raviart element (CR) [4] for simplicial meshes.

For all these space discretizations, the degrees of freedom for the pressure and the density (*i.e.* the discrete pressure and density unknowns) are associated to the cells of the mesh $\mathcal{M}$, and are denoted by:

$$\big\{p_K,\ \rho_K,\ K \in \mathcal{M}\big\}.$$

Let us then turn to the degrees of freedom for the velocity (*i.e.* the discrete velocity unknowns).

- **Rannacher-Turek** or **Crouzeix-Raviart** discretizations – The degrees of freedom for the velocity components are located at the center of the faces of the mesh, and we choose the version of the element where they represent the average of the velocity through a face. The set of degrees of freedom reads:

$$\{u_{\sigma,i},\ \sigma \in \mathcal{E},\ 1 \leq i \leq d\}.$$

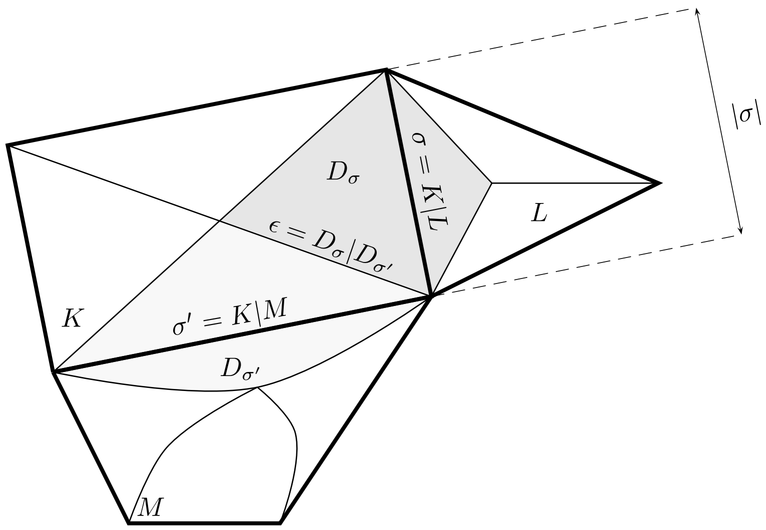

FIG. 2.1. *Primal and dual meshes for the Rannacher-Turek and Crouzeix-Raviart elements.*

- **MAC** discretization – The degrees of freedom for the i^{th} component of the velocity are defined at the centre of the faces $\sigma \in \mathcal{E}^{(i)}$, so the whole set of discrete velocity unknowns reads:

$$\big\{u_{\sigma,i},\ \sigma \in \mathcal{E}^{(i)},\ 1 \le i \le d\big\}.$$

We now introduce a dual mesh, which will be used for the finite volume approximation of the time derivative and convection terms in the momentum balance equation.

- **Rannacher-Turek** or **Crouzeix-Raviart** discretizations – For the RT or CR discretizations, the dual mesh is the same for all the velocity components. When $K \in \mathcal{M}$ is a simplex, a rectangle or a cuboid, for $\sigma \in \mathcal{E}(K)$, we define $D_{K,\sigma}$ as the cone with basis σ and with vertex the mass center of K (see Figure 2.1). We thus obtain a partition of K in m sub-volumes, where m is the number of faces of the mesh, each sub-volume having the same measure $|D_{K,\sigma}| = |K|/m$. We extend this definition to general quadrangles and hexahedra, by supposing that we have built a partition still of equal-volume sub-cells, and with the same connectivities. Note that this is of course always possible, but that such a volume $D_{K,\sigma}$ may be no longer a cone; indeed, if K is far from a parallelogram, it may not be possible to build a cone having σ as basis, the opposite vertex lying in K and a volume equal to $|K|/m$. The volume $D_{K,\sigma}$ is referred to as the half-diamond cell associated to K and σ.
  For $\sigma \in \mathcal{E}_{\rm int}$, $\sigma = K|L$, we now define the diamond cell D_σ associated to σ by $D_\sigma = D_{K,\sigma} \cup D_{L,\sigma}$; for an external face $\sigma \in \mathcal{E}_{\rm ext} \cap \mathcal{E}(K)$, D_σ is just the same volume as $D_{K,\sigma}$.
- **MAC** discretization – For the MAC scheme, the dual mesh depends on the component of the velocity. For each component, the MAC dual mesh only differs from the RT or CR dual mesh by the choice of the half-diamond cell, which, for $K \in \mathcal{M}$ and $\sigma \in \mathcal{E}(K)$, is now the rectangle or rectangular parallelepiped of basis σ and of measure $|D_{K,\sigma}| = |K|/2$.

We denote by $|D_\sigma|$ the measure of the dual cell D_σ, and by $\epsilon = D_\sigma|D_{\sigma'}$ the face separating two diamond cells D_σ and $D_{\sigma'}$.

Finally, we need to deal with the impermeability (*i.e.* $\boldsymbol{u} \cdot \boldsymbol{n} = 0$) boundary condition. Since the velocity unknowns lie on the boundary (and not inside the cells), these conditions are taken into account in the definition of the discrete spaces. To avoid technicalities in the expression of the schemes, we suppose throughout this paper that the boundary is *a.e.* normal to a coordinate axis, (even in the case of the RT or CR discretizations), which allows to simply set to zero the corresponding velocity unknowns:

$$\text{for } i = 1, \ldots, d, \ \forall \sigma \in \mathcal{E}^{(i)}_{\rm ext}, \qquad u_{\sigma,i} = 0. \tag{2.1}$$

Therefore, there are no degrees of freedom for the velocity on the boundary for the MAC scheme, and there are only $d-1$ degrees of freedom on each boundary face for the CR and RT discretizations, which depend on the orientation of the face. In order to be able to write a unique expression of the discrete equations for both MAC and CR/RT schemes, we introduce the set of faces $\mathcal{E}^{(i)}_{\mathcal{S}}$ associated to the degrees of freedom of each component of the velocity ($\mathcal{S}$ stands for "scheme"):

$$\mathcal{E}^{(i)}_{\mathcal{S}} = \left| \begin{array}{l} \mathcal{E}^{(i)} \setminus \mathcal{E}^{(i)}_{\rm ext} \text{ for the MAC scheme,} \\ \mathcal{E} \setminus \mathcal{E}^{(i)}_{\rm ext} \text{ for the CR or RT schemes.} \end{array} \right.$$

For both schemes, we define $\tilde{\mathcal{E}}^{(i)}$, for $1 \leq i \leq d$, as the set of faces of the dual mesh associated to the i^{th} component of the velocity. For the RT or CR discretizations, the sets $\tilde{\mathcal{E}}^{(i)}$ does not depend on the component (*i.e.* of i), up to the elimination of some unknowns (and so some dual cells and, finally, some external faces) to take the boundary conditions into account. For the MAC scheme, $\tilde{\mathcal{E}}^{(i)}$ depends on i; note that each face of $\tilde{\mathcal{E}}^{(i)}$ is perpendicular to a unit vector of the canonical basis of $\mathbb{R}^d$, but not necessarily to the i^{th} one.

Extension to general domains (of course, with the CR or RT discretizations) may be obtained by redefining, through linear combinations, the degrees of freedom at the external faces, so as to introduce the normal velocity as a new degree of freedom.

3. The scheme. Let us consider a partition $0 = t_0 < t_1 < \ldots < t_N = T$ of the time interval $(0, T)$, which we suppose uniform for the sake of simplicity, and let $\delta t = t_{n+1} - t_n$ for $n = 0, 1, \ldots, N-1$ be the (constant) time step. We consider an explicit-in-time scheme, which reads in its fully discrete form, for $0 \leq n \leq N-1$:

$$\forall K \in \mathcal{M}, \qquad \frac{|K|}{\delta t}(\rho^{n+1}_K - \rho^n_K) + \sum_{\sigma \in \mathcal{E}(K)} F^n_{K,\sigma} = 0, \tag{3.1a}$$

$$\forall K \in \mathcal{M}, \qquad p^{n+1}_K = \wp(\rho^{n+1}_K) = (\rho^{n+1}_K)^\gamma, \tag{3.1b}$$

For $1 \leq i \leq d$, $\forall \sigma \in \mathcal{E}^{(i)}_{\mathcal{S}}$,

$$\frac{|D_\sigma|}{\delta t}(\rho^{n+1}_{D_\sigma} u^{n+1}_{\sigma,i} - \rho^n_{D_\sigma} u^n_{\sigma,i}) + \sum_{\epsilon \in \tilde{\mathcal{E}}(D_\sigma)} F^n_{\sigma,\epsilon} u^n_{\epsilon,i} + |D_\sigma| \, (\boldsymbol{\nabla} p)^{n+1}_{\sigma,i} = 0, \tag{3.1c}$$

where the terms introduced for each discrete equation are defined herafter.

Equation (3.1a) is obtained by the discretization of the mass balance equation (1.1a) over the primal mesh, and $F^n_{K,\sigma}$ stands for the mass flux across σ outward K, which, because of the impermeability condition, vanishes on external faces and is given on the internal faces by:

$$\forall \sigma = K|L \in \mathcal{E}_{\rm int}, \qquad F^n_{K,\sigma} = |\sigma| \ \rho^n_\sigma \ u^n_{K,\sigma}, \tag{3.2}$$

where $u^n_{K,\sigma}$ is an approximation of the normal velocity to the face σ outward K, defined by:

$$u^n_{K,\sigma} = \left| \begin{array}{l} u^n_{\sigma,i} \ \boldsymbol{e}^{(i)} \cdot \boldsymbol{n}_{K,\sigma} \text{ for } \sigma \in \mathcal{E}^{(i)} \text{ in the MAC case,} \\ \\ \boldsymbol{u}^n_\sigma \cdot \boldsymbol{n}_{K,\sigma} \text{ in the CR and RT cases,} \end{array} \right. \tag{3.3}$$

where $\boldsymbol{e}^{(i)}$ denotes the i-th vector of the orthonormal basis of $\mathbb{R}^d$. The density at the face $\sigma = K|L$ is approximated by the upwind technique:

$$\rho_\sigma^n = \left| \begin{array}{ll} \rho_K^n & \text{if } u_{K,\sigma}^n \geq 0, \\ \rho_L^n & \text{otherwise.} \end{array} \right. \tag{3.4}$$

We now turn to the discrete momentum balance (3.1c), which is obtained by discretizing the momentum balance equation (1.1b) on the dual cells associated to the faces of the mesh. The first task is to define the the values $\rho_{D_\sigma}^{n+1}$ and $\rho_{D_\sigma}^n$, which approximate the density over the dual cell D_σ at time t^{n+1} and t^n respectively, and the discrete mass flux through the dual face ϵ outward D_σ, denoted by $F_{\sigma,\epsilon}^n$; the guideline for their construction is that a finite volume discretization of the mass balance equation over the diamond cells, of the form

$$\forall \sigma \in \mathcal{E}, \qquad \frac{|D_\sigma|}{\delta t} \, (\rho_{D_\sigma}^{n+1} - \rho_{D_\sigma}^n) + \sum_{\epsilon \in \tilde{\mathcal{E}}(D_\sigma)} F_{\sigma,\epsilon}^n = 0, \tag{3.5}$$

must hold in order to be able to derive a discrete kinetic energy balance (see Section 4 below). The density on the dual cells is given by the following weighted average:

$$\begin{aligned} &\text{for } \sigma = K|L \in \mathcal{E}_{\rm int}, \text{ for } k = n \text{ and } k = n+1, \\ &\qquad\qquad |D_\sigma| \; \rho_{D_\sigma}^k = |D_{K,\sigma}| \; \rho_K^k + |D_{L,\sigma}| \; \rho_L^k. \end{aligned} \tag{3.6}$$

For the MAC scheme, the flux on a dual face which is located on two primal faces is the mean value of the sum of fluxes on the two primal faces, and the flux of a dual face located between two primal faces is again the mean value of the sum of fluxes on the two primal faces [14]. In the case of the CR and RT schemes, for a dual face ϵ included in the primal cell K, this flux is computed as a linear combination (with constant coefficients, *i.e.* independent of the cell) of the mass fluxes through the faces of K, *i.e.* the quantities $(F_{K,\sigma}^n)_{\sigma \in \mathcal{E}(K)}$ appearing in the discrete mass balance (3.1a). We refer to [1, 6] for a detailed construction of this approximation. Let us remark that a dual face lying on the boundary is then also a primal face, and the flux across this face is zero. Therefore, the values $u_{\epsilon,i}^{n+1}$ are only needed at the internal dual faces, and are upwinded:

$$\text{for } \epsilon = D_\sigma | D_{\sigma'}, \quad u_{\epsilon,i}^n = \left| \begin{array}{l} u_{\sigma,i}^n \text{ if } F_{\sigma,\epsilon}^n \geq 0, \\ u_{\sigma',i}^n \text{ otherwise.} \end{array} \right. \tag{3.7}$$

The last term $(\boldsymbol{\nabla} p)_{\sigma,i}^{n+1}$ stands for the i-th component of the discrete pressure gradient at the face σ. The gradient operator is built as the transpose of the discrete operator for the divergence of the velocity, the discretization of which is based on the primal mesh. Let us denote the divergence of $\boldsymbol{u}^{n+1}$ over $K \in \mathcal{M}$ by $(\mathrm{div}\boldsymbol{u})_K^{n+1}$; its natural approximation reads:

$$\text{for } K \in \mathcal{M}, \qquad (\mathrm{div}\boldsymbol{u})_K^{n+1} = \frac{1}{|K|} \sum_{\sigma \in \mathcal{E}(K)} |\sigma| \; u_{K,\sigma}^{n+1}. \tag{3.8}$$

Consequently, we choose the components of the pressure gradient as:

$$\text{for } \sigma = K|L \in \mathcal{E}_{\rm int}, \qquad (\boldsymbol{\nabla} p)_{\sigma,i}^{n+1} = \frac{|\sigma|}{|D_\sigma|} \; (p_L^{n+1} - p_K^{n+1}) \; \boldsymbol{n}_{K,\sigma} \cdot \boldsymbol{e}^{(i)}, \tag{3.9}$$

in order that the following duality relation (with respect to the L^2 inner product) be satisfied:

$$\sum_{K \in \mathcal{M}} |K| \; p_K^{n+1} \; (\mathrm{div}\boldsymbol{u})_K^{n+1} + \sum_{i=1}^d \sum_{\sigma \in \mathcal{E}_{\mathcal{S}}^{(i)}} |D_\sigma| \; u_{\sigma,i}^{n+1} \; (\boldsymbol{\nabla} p)_{\sigma,i}^{n+1} = 0. \tag{3.10}$$

Note that, because of the impermeability boundary conditions, the discrete gradient is not defined at the external faces.

Finally, the initial approximations for ρ and $\boldsymbol{u}$ are given by the average of the initial conditions ρ_0 and $\boldsymbol{u}_0$ on the primal and dual cells respectively:

$$\begin{aligned} &\forall K \in \mathcal{M}, \quad \rho^0_K = \frac{1}{|K|} \int_K \rho_0(\boldsymbol{x}) \,\mathrm{d}\boldsymbol{x}, \\ &\text{for } 1 \leq i \leq d,\ \forall \sigma \in \mathcal{E}_{\mathcal{S}}^{(i)}, \quad u^0_{\sigma,i} = \frac{1}{|D_\sigma|} \int_{D_\sigma} (\boldsymbol{u}_0(\boldsymbol{x}))_i \,\mathrm{d}\boldsymbol{x}. \end{aligned} \tag{3.11}$$

The following positivity result is a classical consequence of the upwind choice in the mass balance equation.

LEMMA 3.1 (Positivity of the density). *Let ρ^0 be given by (3.11). Then, since ρ_0 is assumed to be a positive function, $\rho^0 > 0$ and, under the CFL condition:*

$$\delta t \leq \frac{|K|}{\sum_{\sigma \in \mathcal{E}(K)} |\sigma| \ \max(u^n_{K,\sigma}, 0)}, \quad \forall K \in \mathcal{M} \text{ and for } 0 \leq n \leq N-1, \tag{3.12}$$

the solution to the scheme satisfies $\rho^n > 0$, for $1 \leq n \leq N$.

4. Discrete kinetic energy and elastic potential balances. We begin by deriving a discrete kinetic energy balance equation, as was already done in [12] in the implicit and fractional time step cases. Equation (4.1) is a discrete analogue of Equation (1.2), with an upwind discretization of the convection term.

LEMMA 4.1 (Discrete kinetic energy balance).
A solution to the system (3.1) satisfies the following equality, for $1 \leq i \leq d$, $\sigma \in \mathcal{E}_{\mathcal{S}}^{(i)}$ and $0 \leq n \leq N-1$:

$$\begin{aligned} \frac{1}{2} \frac{|D_\sigma|}{\delta t} \Big[\rho^{n+1}_{D_\sigma} (u^{n+1}_{\sigma,i})^2 - \rho^n_{D_\sigma} (u^n_{\sigma,i})^2\Big] + \frac{1}{2} \sum_{\epsilon \in \tilde{\mathcal{E}}(D_\sigma)} F^n_{\sigma,\epsilon}\ (u^n_{\epsilon,i})^2 \\ + |D_\sigma|\, (\boldsymbol{\nabla} p)^{n+1}_{\sigma,i}\ u^{n+1}_{\sigma,i} = -R^{n+1}_{\sigma,i}, \end{aligned} \tag{4.1}$$

with:

$$\begin{aligned} R^{n+1}_{\sigma,i} = \frac{1}{2} \frac{|D_\sigma|}{\delta t} \rho^{n+1}_{D_\sigma} (u^{n+1}_{\sigma,i} - u^n_{\sigma,i})^2 + \frac{1}{2} \sum_{\epsilon = D_\sigma | D_{\sigma'} \in \tilde{\mathcal{E}}(D_\sigma)} (F^n_{\sigma,\epsilon})^- (u^n_{\sigma',i} - u^n_{\sigma,i})^2 \\ - \sum_{\epsilon = D_\sigma | D_{\sigma'} \in \tilde{\mathcal{E}}(D_\sigma)} (F^n_{\sigma,\epsilon})^- (u^n_{\sigma',i} - u^n_{\sigma,i})\,(u^{n+1}_{\sigma,i} - u^n_{\sigma,i}), \end{aligned} \tag{4.2}$$

where, for $a \in \mathbb{R}$, $a^- \geq 0$ is defined by $a^- = -\min(a, 0)$. This remainder term is non-negative under the following CFL condition:

$$\forall \sigma \in \mathcal{E}_{\mathcal{S}}^{(i)}, \qquad \delta t \leq \frac{|D_\sigma|\ \rho^{n+1}_{D_\sigma}}{\sum_{\epsilon \in \tilde{\mathcal{E}}(D_\sigma)} (F^n_{\sigma,\epsilon})^-}. \tag{4.3}$$

Proof. The proof of this lemma is obtained by multiplying the (i^{th} component of the) momentum balance equation (3.1c) associated to the face σ by the unknown $u^{n+1}_{\sigma,i}$, and invoking Lemma A.2 of the appendix. ☐

Similarly, the solution to the scheme (3.1) satisfies a discrete version of the elastic potential identity (1.4), which we now state.

LEMMA 4.2 (Discrete potential balance). *Let $\mathcal{H}$ be defined by (1.3). A solution to the system (3.1) satisfies the following equality, for $K \in \mathcal{M}$ and $0 \leq n \leq N-1$:*

$$\frac{|K|}{\delta t} \Big[\mathcal{H}(\rho^{n+1}_K) - \mathcal{H}(\rho^n_K)\Big] + \sum_{\sigma \in \mathcal{E}(K)} |\sigma|\ \mathcal{H}(\rho^n_\sigma)\ u^n_{K,\sigma} + |K|\ p^n_K (\mathrm{div} \boldsymbol{u}^n)_K = -R^{n+1}_K. \tag{4.4}$$

In this relation, the remainder term is defined by:

$$R_K^{n+1} = \frac{1}{2}\frac{|K|}{\delta t}\,\mathcal{H}''(\overline{\rho}_{K,1}^n)\,(\rho_K^{n+1}-\rho_K^n)^2 + \frac{1}{2}\sum_{\sigma=K|L\in\mathcal{E}(K)} |\sigma|\,(u_{K,\sigma}^n)^-\,\mathcal{H}''(\overline{\rho}_\sigma^n)\,(\rho_K^n-\rho_L^n)^2 + \sum_{\sigma\in\mathcal{E}(K)} |\sigma|\,u_{K,\sigma}^n\,\mathcal{H}''(\overline{\rho}_{K,2}^n)\,\rho_\sigma^n\,(\rho_K^{n+1}-\rho_K^n), \quad (4.5)$$

with $\overline{\rho}_{K,1}^n$, $\overline{\rho}_{K,2}^n \in [\![\rho_K^{n+1},\rho_K^n]\!]$, *and* $\overline{\rho}_\sigma^n \in [\![\rho_K^n,\rho_\sigma^n]\!]$ *for all* $\sigma\in\mathcal{E}(K)$, *where, for* $a,\ b\in\mathbb{R}$, *we denote by* $[\![a,b]\!]$ *the interval* $[\![a,b]\!]=\{\theta a+(1-\theta)b,\ \theta\in[0,1]\}$.

Proof. The proof of this lemma is obtain by multiplying the discrete mass balance equation (3.1a) by $\mathcal{H}'(\rho_K^{n+1})$ and invoking Lemma A.1 of the appendix. ∎

Unfortunately, it does not seem that $R_K^{n+1}\geq 0$ in any case, and so we are not able to prove a discrete counterpart of the total entropy estimate (1.8), which would yield a stability estimate for the scheme. However, under a condition for a time step which is only slightly more restrictive than a CFL-condition, and under some stability assumptions for the solutions to the scheme, we are able to show that the possible non-positive part of this remainder term tends to zero in $\mathrm{L}^1(\Omega\times(0,T))$ with the space and time steps, which allows to conclude, in the 1D case, that a convergent sequence of solutions satisfies the entropy inequality (1.7): this is the result stated in Lemma 5.3 below.

5. Passing to the limit in the scheme. The objective of this section is to show, in the one dimensional case, that if a sequence of solutions is controlled in suitable norms and converges to a limit, this latter necessarily satisfies a (part of the) weak formulation of the continuous problem.

The 1D version of the scheme which is studied in this section may be obtained from Scheme (3.1) by taking the MAC variant of the scheme, using only one horizontal stripe of grid cells, supposing that the vertical component of the velocity (the degrees of freedom of which are located on the top and bottom boundaries) vanishes, and that the measure of the vertical faces is equal to 1. For the sake of readability, however, we completely rewrite this 1D scheme, and, to this purpose, we first introduce some adaptations of the notations to the one dimensional case. For any face $\sigma\in\mathcal{E}$, let x_σ be its abscissa. For $K\in\mathcal{M}$, we denote by h_K its length (so $h_K=|K|$); when we write $K=[\sigma\sigma']$, this means that either $K=(x_\sigma,x_{\sigma'})$ or $K=(x_{\sigma'},x_\sigma)$; if we need to specify the order, *i.e.* $K=(x_\sigma,x_{\sigma'})$ with $x_\sigma<x_{\sigma'}$, then we write $K=[\overrightarrow{\sigma\sigma'}]$. For an interface $\sigma=K|L$ between two cells K and L, we define $h_\sigma=(h_K+h_L)/2$, so, by definition of the dual mesh, $h_\sigma=|D_\sigma|$. If we need to specify the order of the cells K and L, say K is left of L, then we write $\sigma=\overrightarrow{K|L}$. With these notations, the explicit scheme (3.1) may be written as follows in the one dimensional setting:

$$\begin{aligned} &\forall K\in\mathcal{M}, && \rho_K^0=\frac{1}{|K|}\int_K \rho_0(x)\,\mathrm{d}x,\\ &\forall\sigma\in\mathcal{E}_{\rm int}, && u_\sigma^0=\frac{1}{|D_\sigma|}\int_{D_\sigma} u_0(x)\,\mathrm{d}x, \end{aligned} \quad (5.1a)$$

$\forall K=[\overrightarrow{\sigma\sigma'}]\in\mathcal{M}$,

$$\frac{|K|}{\delta t}(\rho_K^{n+1}-\rho_K^n)+F_{\sigma'}^n-F_\sigma^n=0, \quad (5.1b)$$

$$\forall K\in\mathcal{M},\qquad p_K^{n+1}=\wp(\rho_K^{n+1})=(\rho_K^{n+1})^\gamma, \quad (5.1c)$$

$\forall\sigma=\overrightarrow{K|L}\in\mathcal{E}_{\rm int}$,

$$\frac{|D_\sigma|}{\delta t}(\rho_{D_\sigma}^{n+1}u_\sigma^{n+1}-\rho_{D_\sigma}^n u_\sigma^n)+F_L^n u_L^n-F_K^n u_K^n+p_L^{n+1}-p_K^{n+1}=0. \quad (5.1d)$$

The mass flux in the discrete mass balance equation is given, for $\sigma \in \mathcal{E}_{\rm int}$, by $F_\sigma^n = \rho_\sigma^n u_\sigma^n$, where the upwind approximation for the density at the face, ρ_σ^n, is defined by (3.4). In the momentum balance equation, the density associated to the dual cell D_σ, with $\sigma = K|L$, reads

$$\text{for } k = n \text{ and } k = n+1, \quad \rho_{D_\sigma}^k = \frac{1}{2\,|D_\sigma|}\,(|K|\,\rho_K^k + |L|\,\rho_L^k), \tag{5.2}$$

and the application of the procedure described in Section 3 yields, for the mass fluxes at the dual face located at the center of the mesh $K = [\overrightarrow{\sigma\sigma'}]$:

$$F_K^n = \frac{1}{2}\,(F_\sigma^n + F_{\sigma'}^n). \tag{5.3}$$

The approximation of the velocity at this face is upwind: $u_K^n = u_\sigma^n$ if $F_K^n \geq 0$ and $u_K^n = u_{\sigma'}^n$ otherwise.

Let a sequence of discretizations $(\mathcal{M}^{(m)},\ \delta t^{(m)})_{m\in\mathbb{N}}$ be given. We define the size $h^{(m)}$ of the mesh $\mathcal{M}^{(m)}$ by $h^{(m)} = \sup_{K\in\mathcal{M}^{(m)}} h_K$. Let $\rho^{(m)}$, $p^{(m)}$ and $u^{(m)}$ be the solution given by the scheme (5.1) with the mesh $\mathcal{M}^{(m)}$ and the time step $\delta t^{(m)}$. To the discrete unknowns, we associate piecewise constant functions on time intervals and on primal or dual meshes, so the density $\rho^{(m)}$, the pressure $p^{(m)}$ and the velocity $u^{(m)}$ are defined almost everywhere on $\Omega \times (0,T)$ by:

$$\begin{aligned} \rho^{(m)}(x,t) &= \sum_{n=0}^{N-1} \sum_{K\in\mathcal{M}} (\rho^{(m)})_K^n\, \mathcal{X}_K(x)\, \mathcal{X}_{[n,n+1)}(t), \\ p^{(m)}(x,t) &= \sum_{n=0}^{N-1} \sum_{K\in\mathcal{M}} (p^{(m)})_K^n\, \mathcal{X}_K(x)\, \mathcal{X}_{[n,n+1)}(t), \\ u^{(m)}(x,t) &= \sum_{n=0}^{N-1} \sum_{\sigma\in\mathcal{E}} (u^{(m)})_\sigma^n\, \mathcal{X}_{D_\sigma}(x)\, \mathcal{X}_{[n,n+1)}(t), \end{aligned} \tag{5.4}$$

where $\mathcal{X}_K$, $\mathcal{X}_{D_\sigma}$ and $\mathcal{X}_{[n,n+1)}$ stand for the characteristic function of the intervals K, D_σ and $[t^n, t^{n+1})$ respectively.

For discrete functions q and v defined on the primal and dual mesh, respectively, we define a discrete $\mathrm{L}^1((0,T);\mathrm{BV}(\Omega))$ norm by:

$$\|q\|_{\mathcal{T},x,\mathrm{BV}} = \sum_{n=0}^{N} \delta t \sum_{\sigma=K|L\in\mathcal{E}_{\rm int}} |q_L^n - q_K^n|, \quad \|v\|_{\mathcal{T},x,\mathrm{BV}} = \sum_{n=0}^{N} \delta t \sum_{\epsilon=D_\sigma|D_{\sigma'}\in\tilde{\mathcal{E}}_{\rm int}} |v_{\sigma'}^n - v_\sigma^n|,$$

and a discrete $\mathrm{L}^1(\Omega;\mathrm{BV}((0,T)))$ norm by:

$$\|q\|_{\mathcal{T},t,\mathrm{BV}} = \sum_{K\in\mathcal{M}} |K| \sum_{n=0}^{N-1} |q_K^{n+1} - q_K^n|, \qquad \|v\|_{\mathcal{T},t,\mathrm{BV}} = \sum_{\sigma\in\mathcal{E}} |D_\sigma| \sum_{n=0}^{N-1} |v_\sigma^{n+1} - v_\sigma^n|.$$

For the consistency result that we are seeking (Theorem 5.2 below), we have to assume that a sequence of discrete solutions $\big(\rho^{(m)}, p^{(m)}, u^{(m)}\big)_{m\in\mathbb{N}}$ satisfies $\rho^{(m)} > 0$ and $p^{(m)} > 0$, $\forall m \in \mathbb{N}$ (which may be a consequence of the fact that the CFL stability condition (3.12) is satisfied), and is uniformly bounded in $\mathrm{L}^\infty((0,T)\times\Omega)^3$, *i.e.*:

$$0 < (\rho^{(m)})_K^n \leq C,\ 0 < (p^{(m)})_K^n \leq C, \quad \forall K \in \mathcal{M}^{(m)},\ \text{for } 0 \leq n \leq N^{(m)},\ \forall m \in \mathbb{N}, \tag{5.5}$$

and

$$|(u^{(m)})_\sigma^n| \leq C, \quad \forall \sigma \in \mathcal{E}^{(m)},\ \text{for } 0 \leq n \leq N^{(m)},\ \forall m \in \mathbb{N}, \tag{5.6}$$

where C is a positive real number. Note that, by definition of the initial conditions of the scheme, these inequalities imply that the functions ρ_0 and u_0 belong to $\mathrm{L}^\infty(\Omega)$.

We also have to assume that a sequence of discrete solutions satisfies the following uniform bounds in the discrete BV-norms:

$$\|\rho^{(m)}\|_{\mathcal{T},x,\mathrm{BV}} + \|u^{(m)}\|_{\mathcal{T},x,\mathrm{BV}} \leq C, \quad \forall m \in \mathbb{N}. \tag{5.7}$$

We are not able to prove the estimates (5.5)–(5.7) for the solutions of the scheme; however, such inequalities are satisfied by the "interpolates" (for instance, by taking the cell average) of the solution to a Riemann problem, and are observed in computations (of course, as far as possible, *i.e.* in a limited number of cases and with a limited sequence of meshes and time steps).

A weak solution to the continuous problem satisfies, for any $\varphi \in \mathrm{C}_c^\infty\big(\Omega \times [0,T)\big)$:

$$-\int_0^T \int_\Omega \Big[\rho\, \partial_t \varphi + \rho\, u\, \partial_x \varphi\Big]\, \mathrm{d}x\, \mathrm{d}t - \int_\Omega \rho_0(x)\, \varphi(x,0)\, \mathrm{d}x = 0, \tag{5.8a}$$

$$-\int_0^T \int_\Omega \Big[\rho\, u\, \partial_t \varphi + (\rho\, u^2 + p)\, \partial_x \varphi\Big]\, \mathrm{d}x\, \mathrm{d}t - \int_\Omega \rho_0(x)\, u_0(x)\, \varphi(x,0)\, \mathrm{d}x = 0, \tag{5.8b}$$

$$p = \rho^\gamma. \tag{5.8c}$$

Note that these relations are not sufficient to define a weak solution to the problem, since they do not imply anything about the boundary conditions. However, they allow to derive the Rankine-Hugoniot conditions; hence if we show that they are satisfied by the limit of a sequence of solutions to the scheme, this implies, loosely speaking, that *the scheme computes correct shocks* (*i.e.* shocks where the jumps of the unknowns and of the fluxes are linked to the shock speed by Rankine-Hugoniot conditions). This is the result we are seeking and which we state in Theorem 5.2. In order to prove this theorem, we need some definitions of interpolates of regular test functions on the primal and dual mesh.

Definition 5.1 (Interpolates on one-dimensional meshes). *Let Ω be an open bounded interval of $\mathbb{R}$, let $\varphi \in \mathrm{C}_c^\infty(\Omega \times [0,T))$, and let $\mathcal{M}$ be a mesh over Ω. For $0 \leq n \leq N$ and $K \in \mathcal{M}$, we set $\varphi_K^n = \varphi(x_K, t^n)$, with x_K the mass center of K. The interpolate $\varphi_\mathcal{M}$ of φ on the primal mesh $\mathcal{M}$ is defined by:*

$$\varphi_\mathcal{M}(x,0) = \sum_{K \in \mathcal{M}} \varphi_K^0\, \mathcal{X}_K \ \textit{and, for } t > 0,\ \varphi_\mathcal{M} = \sum_{n=0}^{N-1} \sum_{K \in \mathcal{M}} \varphi_K^{n+1}\, \mathcal{X}_K\, \mathcal{X}_{(t^n, t^{n+1}]}. \tag{5.9}$$

The time discrete derivative of $\varphi_\mathcal{M}$ is given by:

$$\eth_t \varphi_\mathcal{M} = \sum_{n=0}^{N-1} \sum_{K \in \mathcal{M}} \frac{\varphi_K^{n+1} - \varphi_K^n}{\delta t}\, \mathcal{X}_K\ \mathcal{X}_{(t^n, t^{n+1}]}, \tag{5.10}$$

and its space discrete derivative by:

$$\eth_x \varphi_\mathcal{M} = \sum_{n=0}^{N-1} \sum_{\sigma = \overrightarrow{K|L} \in \mathcal{E}_{\mathrm{int}}} \frac{\varphi_L^{n+1} - \varphi_K^{n+1}}{h_\sigma}\, \mathcal{X}_{D_\sigma}\ \mathcal{X}_{(t^n, t^{n+1}]}. \tag{5.11}$$

For $0 \leq n \leq N$ and $\sigma \in \mathcal{E}$, we set $\varphi_\sigma^n = \varphi(x_\sigma, t^n)$. Then $\varphi_\mathcal{E}$, the interpolate of φ on the dual mesh, is defined by:

$$\varphi_\mathcal{E}(x,0) = \sum_{\sigma \in \mathcal{E}} \varphi_\sigma^0\ \mathcal{X}_{D_\sigma} \quad \textit{and, for } t > 0,\ \varphi_\mathcal{E} = \sum_{n=0}^{N-1} \sum_{\sigma \in \mathcal{E}} \varphi_\sigma^{n+1}\ \mathcal{X}_{D_\sigma}\ \mathcal{X}_{(t^n, t^{n+1}]}. \tag{5.12}$$

We also define the time and space discrete derivatives of this function by:

$$\begin{aligned}
\eth_t \varphi_\mathcal{E} &= \sum_{n=0}^{N-1} \sum_{\sigma \in \mathcal{E}} \frac{\varphi_\sigma^{n+1} - \varphi_\sigma^n}{\delta t}\ \mathcal{X}_{D_\sigma}\ \mathcal{X}_{(t^n, t^{n+1}]}, \\
\eth_x \varphi_\mathcal{E} &= \sum_{n=0}^{N-1} \sum_{K = [\overrightarrow{\sigma\sigma'}] \in \mathcal{M}} \frac{\varphi_{\sigma'}^{n+1} - \varphi_\sigma^{n+1}}{h_K}\ \mathcal{X}_K\ \mathcal{X}_{(t^n, t^{n+1}]}.
\end{aligned} \tag{5.13}$$

We are now in position to state the following result.

Theorem 5.2 (Consistency of the one-dimensional scheme).
Let Ω be an open bounded interval of $\mathbb{R}$. We suppose that the initial data satisfies $\rho_0 \in \mathrm{L}^\infty(\Omega)$ and $u_0 \in \mathrm{L}^\infty(\Omega)$. Let $(\mathcal{M}^{(m)}, \delta t^{(m)})_{m\in\mathbb{N}}$ be a sequence of discretizations such that both the time step $\delta t^{(m)}$ and the size $h^{(m)}$ of the mesh $\mathcal{M}^{(m)}$ tend to zero as $m \to +\infty$, and let $(\rho^{(m)}, p^{(m)}, u^{(m)})_{m\in\mathbb{N}}$ be the corresponding sequence of solutions. We suppose that this sequence satisfies the estimates (5.5)–(5.7) and converges in $\mathrm{L}^r(\Omega\times(0,T))^3$, for $1 \le r < \infty$, to $(\bar\rho, \bar p, \bar u) \in \mathrm{L}^\infty(\Omega\times(0,T))^3$.

Then the limit $(\bar\rho, \bar p, \bar u)$ satisfies the system (5.8).

Proof. It is clear that, with the assumed convergence for the sequence of solutions, the limit satisfies the equation of state. The proof of this theorem is thus obtained by passing to the limit in the scheme for the mass balance equation first, and then for the momentum balance equation.

Mass balance equation – Let $\varphi \in \mathrm{C}_c^\infty(\Omega\times[0,T))$. Let $m\in\mathbb{N}$, $\mathcal{M}^{(m)}$ and $\delta t^{(m)}$ be given. Dropping for short the superscript $^{(m)}$, let $\varphi_\mathcal{M}$ be the interpolate of φ on the primal mesh and let $\eth_t\varphi_\mathcal{M}$ and $\eth_x\varphi_\mathcal{M}$ be its time and space discrete derivatives in the sense of Definition 5.1. Thanks to the regularity of φ, these functions respectively converge in $\mathrm{L}^r(\Omega\times(0,T))$, for $r\ge 1$ (including $r=+\infty$), to φ, $\partial_t\varphi$ and $\partial_x\varphi$ respectively. In addition, $\varphi_\mathcal{M}(\cdot,0)$ (which, for $K\in\mathcal{M}$ and $x\in K$, is equal to $\varphi_K^0 = \varphi(x,0)$) converges to $\varphi(\cdot,0)$ in $\mathrm{L}^r(\Omega)$ for $r\ge 1$. Since the support of φ is compact in $\Omega\times[0,T)$, for m large enough, the interpolate of φ vanishes at the boundary cells and at the last time step(s); hereafter, we systematically assume that we are in this case.

Let us multiply the first equation (3.1a) of the scheme by $\delta t\,\varphi_K^{n+1}$, and sum the result for $0\le n\le N-1$ and $K\in\mathcal{M}$, to obtain $T_1^{(m)} + T_2^{(m)} = 0$ with

$$T_1^{(m)} = \sum_{n=0}^{N-1}\sum_{K\in\mathcal{M}} |K|(\rho_K^{n+1}-\rho_K^n)\,\varphi_K^{n+1}, \qquad T_2^{(m)} = \sum_{n=0}^{N-1}\delta t \sum_{K=[\overrightarrow{\sigma\sigma'}]\in\mathcal{M}} (F_{\sigma'}^n - F_\sigma^n)\,\varphi_K^{n+1}.$$

Reordering the sums in $T_1^{(m)}$ yields:

$$T_1^{(m)} = -\sum_{n=0}^{N-1}\delta t\sum_{K\in\mathcal{M}} |K|\,\rho_K^n\,\frac{\varphi_K^{n+1}-\varphi_K^n}{\delta t} - \sum_{K\in\mathcal{M}} |K|\,\rho_K^0\,\varphi_K^0,$$

so that:

$$T_1^{(m)} = -\int_0^T\int_\Omega \rho^{(m)}\eth_t\varphi_\mathcal{M}\,\mathrm{d}x\,\mathrm{d}t - \int_\Omega (\rho^{(m)})^0(x)\;\varphi_\mathcal{M}(x,0)\,\mathrm{d}x.$$

The boundedness of ρ_0 and the definition (5.1a) of the initial conditions for the scheme ensures that the sequence $((\rho^{(m)})^0)_{m\in\mathbb{N}}$ converges to ρ_0 in $\mathrm{L}^r(\Omega)$ for $r\ge 1$. Since, by assumption, the sequence of discrete solutions and of the interpolate time derivatives converge in $\mathrm{L}^r\big(\Omega\times(0,T)\big)$ for $r\ge 1$, we thus obtain:

$$\lim_{m\to+\infty} T_1^{(m)} = -\int_0^T\int_\Omega \bar\rho\,\partial_t\varphi\,\mathrm{d}x\,\mathrm{d}t - \int_\Omega \rho_0(x)\,\varphi(x,0)\,\mathrm{d}x.$$

Using the expression of the mass flux F_σ^n and reordering the sums in $T_2^{(m)}$, we get, remarking that $|D_\sigma| = h_\sigma$:

$$T_2^{(m)} = -\sum_{n=0}^{N-1}\delta t\sum_{\sigma=\overrightarrow{K|L}\in\mathcal{E}} |D_\sigma|\;\rho_\sigma^n u_\sigma^n\;\frac{\varphi_L^{n+1}-\varphi_K^{n+1}}{h_\sigma}.$$

Since $|D_\sigma| = (|K| + |L|)/2$ and ρ^n_σ is the upwind approximation of ρ^n at the face σ, we can rewrite $T_2^{(m)} = \mathcal{T}_2^{(m)} + \mathcal{R}_2^{(m)}$ with

$$\mathcal{T}_2^{(m)} = -\sum_{n=0}^{N-1} \delta t \sum_{\sigma=\overrightarrow{K|L}\in\mathcal{E}} \Big(\frac{|K|}{2}\rho^n_K + \frac{|L|}{2}\rho^n_L\Big) u^n_\sigma \; \frac{\varphi^{n+1}_L - \varphi^{n+1}_K}{h_\sigma},$$

$$\mathcal{R}_2^{(m)} = -\sum_{n=0}^{N-1} \delta t \sum_{\sigma=\overrightarrow{K|L}\in\mathcal{E}} (\rho^n_K - \rho^n_L) \, \Big[\frac{|K|}{2}(u^n_\sigma)^- + \frac{|L|}{2}(u^n_\sigma)^+\Big] \; \frac{\varphi^{n+1}_L - \varphi^{n+1}_K}{h_\sigma},$$

where, for $a \in \mathbb{R}$, $a^+ = \max(a, 0)$ and $a^- = -\min(a, 0)$ (so $a = a^+ - a^-$). We have, for the term $\mathcal{T}_2^{(m)}$:

$$\mathcal{T}_2^{(m)} = -\int_0^T \int_\Omega \rho^{(m)} u^{(m)} \eth_x \varphi_{\mathcal{M}} \, \mathrm{d}x \, \mathrm{d}t$$

and therefore

$$\lim_{m\to+\infty} \mathcal{T}_2^{(m)} = -\int_0^T \int_\Omega \bar\rho \, \bar u \, \partial_x \varphi \, \mathrm{d}x \, \mathrm{d}t.$$

The remainder term $\mathcal{R}_2^{(m)}$ is bounded as follows, with $C_\varphi = \|\partial_x \varphi\|_{\mathrm{L}^\infty(\Omega\times(0,T))}$:

$$\begin{aligned} |\mathcal{R}_2^{(m)}| \le C_\varphi \sum_{n=0}^{N-1} \delta t \sum_{\sigma=K|L\in\mathcal{E}} |\rho^n_K - \rho^n_L| \; |D_\sigma| \; |u^n_\sigma| \\ \le C_\varphi \; \|u^{(m)}\|_{\mathrm{L}^\infty(\Omega\times(0,T))} \; \|\rho^{(m)}\|_{\mathcal{T},x,\mathrm{BV}} \; h^{(m)}, \end{aligned}$$

and therefore tends to zero when m tends to $+\infty$, by the assumed stability of the solution.

Momentum balance equation – Let $\varphi_\mathcal{E}$, $\eth_t \varphi_\mathcal{E}$ and $\eth_x \varphi_\mathcal{E}$ be the interpolate of φ on the dual mesh and its discrete time and space derivatives, in the sense of Definition 5.1, which converge in $\mathrm{L}^r(\Omega \times (0, T))$, for $r \ge 1$ (including $r = +\infty$), to φ, $\partial_t\varphi$ and $\partial_x\varphi$ respectively. Let us multiply Equation (3.1c) by $\delta t \, \varphi^{n+1}_\sigma$, and sum the result for $0 \le n \le N-1$ and $\sigma \in \mathcal{E}_{\mathrm{int}}$. We obtain $T_1^{(m)} + T_2^{(m)} + T_3^{(m)} = 0$ with

$$T_1^{(m)} = \sum_{n=0}^{N-1} \sum_{\sigma\in\mathcal{E}_{\mathrm{int}}} |D_\sigma| \; (\rho^{n+1}_{D_\sigma} u^{n+1}_\sigma - \rho^n_{D_\sigma} u^n_\sigma) \, \varphi^{n+1}_\sigma,$$

$$T_2^{(m)} = \sum_{n=0}^{N-1} \delta t \sum_{\sigma=\overrightarrow{K|L}\in\mathcal{E}_{\mathrm{int}}} \Big[F^n_L \, u^n_L - F^n_K \, u^n_K\Big] \; \varphi^{n+1}_\sigma,$$

$$T_3^{(m)} = \sum_{n=0}^{N-1} \delta t \sum_{\sigma=\overrightarrow{K|L}\in\mathcal{E}_{\mathrm{int}}} (p^{n+1}_L - p^{n+1}_K) \, \varphi^{n+1}_\sigma.$$

Reordering the sums, we get for $T_1^{(m)}$:

$$T_1^{(m)} = -\sum_{n=0}^{N-1} \delta t \sum_{\sigma\in\mathcal{E}_{\mathrm{int}}} |D_\sigma| \, \rho^n_{D_\sigma} u^n_\sigma \; \frac{\varphi^{n+1}_\sigma - \varphi^n_\sigma}{\delta t} - \sum_{\sigma\in\mathcal{E}_{\mathrm{int}}} |D_\sigma| \, \rho^0_{D_\sigma} \, u^0_\sigma \, \varphi^0_\sigma.$$

Thanks to the definition of the quantity ρ_{D_σ} (namely the fact that $|D_\sigma| \, \rho^n_{D_\sigma} = (|K| \, \rho^n_K + |L| \, \rho^n_L)/2$), we have:

$$T_1^{(m)} = -\int_0^T \int_\Omega \rho^{(m)} \, u^{(m)} \, \eth_t \varphi_\mathcal{E} \, \mathrm{d}x \, \mathrm{d}t - \int_\Omega (\rho^{(m)})^0(x) \; (u^{(m)})^0(x) \; \varphi_\mathcal{E}(x, 0) \, \mathrm{d}x.$$

By the same arguments as for the mass balance equation, we therefore obtain:

$$\lim_{m\to+\infty} T_1^{(m)} = -\int_0^T \int_\Omega \bar\rho\, \bar u\, \partial_t\varphi \,\mathrm{d}x\,\mathrm{d}t - \int_\Omega \rho_0(x)\, u_0(x)\, \varphi(x,0)\,\mathrm{d}x.$$

Let us now turn to $T_2^{(m)}$. Reordering the sums and using the definition of the mass fluxes at the dual faces, we get:

$$\begin{aligned} T_2^{(m)} &= -\sum_{n=0}^{N-1} \delta t \sum_{K=[\overrightarrow{\sigma\sigma'}]\in\mathcal{M}} F_K^n\, u_K^n\ (\varphi_{\sigma'}^{n+1} - \varphi_\sigma^{n+1}) \\ &= -\frac{1}{2}\sum_{n=0}^{N-1} \delta t \sum_{K=[\overrightarrow{\sigma\sigma'}]\in\mathcal{M}} (\rho_\sigma^n u_\sigma^n + \rho_{\sigma'}^n u_{\sigma'}^n)\, u_K^n\ (\varphi_{\sigma'}^{n+1} - \varphi_\sigma^{n+1}). \end{aligned} \tag{5.14}$$

Using the relation

$$\begin{aligned} &\int_0^T \int_\Omega \rho^{(m)}\, (u^{(m)})^2\, \eth_x\varphi_{\mathcal{E}} \,\mathrm{d}x\,\mathrm{d}t \\ &\qquad = \frac{1}{2}\sum_{n=0}^{N-1} \delta t \sum_{K=[\overrightarrow{\sigma\sigma'}]\in\mathcal{M}} \rho_K^n\, \left[(u_\sigma^n)^2 + (u_{\sigma'}^n)^2\right]\ (\varphi_{\sigma'}^{n+1} - \varphi_\sigma^{n+1}), \end{aligned}$$

we can rewrite the term $T_2^{(m)}$ as

$$T_2^{(m)} = -\int_0^T \int_\Omega \rho^{(m)}\, {u^{(m)}}^2\, \eth_x\varphi_{\mathcal{E}} \,\mathrm{d}x\,\mathrm{d}t + \mathcal{R}_2^{(m)},$$

where:

$$\mathcal{R}_2^{(m)} = -\frac{1}{2}\sum_{n=0}^{N-1} \delta t \sum_{K=[\overrightarrow{\sigma\sigma'}]\in\mathcal{M}} \Big[(\rho_\sigma^n u_\sigma^n + \rho_{\sigma'}^n u_{\sigma'}^n)\, u_K^n - \rho_K^n\, \big((u_\sigma^n)^2 + (u_{\sigma'}^n)^2\big)\Big]\ (\varphi_{\sigma'}^{n+1} - \varphi_\sigma^{n+1}).$$

Let us split this latter expression as $\mathcal{R}_2^{(m)} = \mathcal{R}_{21}^{(m)} + \mathcal{R}_{22}^{(m)}$, with:

$$\begin{aligned} \mathcal{R}_{21}^{(m)} &= -\frac{1}{2}\sum_{n=0}^{N-1} \delta t \sum_{K=[\overrightarrow{\sigma\sigma'}]\in\mathcal{M}} u_\sigma^n\ (\rho_\sigma^n u_K^n - \rho_K^n u_\sigma^n)\ (\varphi_{\sigma'}^{n+1} - \varphi_\sigma^{n+1}), \\ \mathcal{R}_{22}^{(m)} &= -\frac{1}{2}\sum_{n=0}^{N-1} \delta t \sum_{K=[\overrightarrow{\sigma\sigma'}]\in\mathcal{M}} u_{\sigma'}^n\ (\rho_{\sigma'}^n u_K^n - \rho_K^n u_{\sigma'}^n)\ (\varphi_{\sigma'}^{n+1} - \varphi_\sigma^{n+1}). \end{aligned}$$

Applying the identity $2(ab-cd) = (a-c)(b+d) + (a+c)(b-d)$, $\forall (a,b,c,d) \in \mathbb{R}^4$, to the term $\rho_\sigma^n u_K^n - \rho_K^n u_\sigma^n$ and using the fact that the quantities $\rho_\sigma^n - \rho_K^n$ and $u_\sigma^n - u_K^n$ are either zero or differences of the density at two neighbouring cells and of the velocity at two neighbouring faces respectively, we obtain for $\mathcal{R}_{21}^{(m)}$:

$$\begin{aligned} |\mathcal{R}_{21}^{(m)}| \leq C_\varphi\ \Big[&\|u^{(m)}\|^2_{\mathrm{L}^\infty(\Omega\times(0,T))}\ \|\rho^{(m)}\|_{\mathcal{T},x,\mathrm{BV}} \\ &+ \|u^{(m)}\|_{\mathrm{L}^\infty(\Omega\times(0,T))}\ \|u^{(m)}\|_{\mathcal{T},x,\mathrm{BV}}\ \|\rho^{(m)}\|_{\mathrm{L}^\infty(\Omega\times(0,T))}\Big]\ h^{(m)}, \end{aligned}$$

where the real number C_φ only depends on φ. Since the same estimate holds for $\mathcal{R}_{22}^{(m)}$, the remainder term $\mathcal{R}_2^{(m)}$ tends to zero when m tends to $+\infty$ and:

$$\lim_{m\to+\infty} T_2^{(m)} = -\int_0^T \int_\Omega \bar\rho\, \bar u^2\, \partial_x\varphi \,\mathrm{d}x\,\mathrm{d}t.$$

Let us finally study $T_3^{(m)}$. Reordering the sums, we obtain $T_3^{(m)} = \mathcal{T}_3^{(m)} + \mathcal{R}_3^{(m)}$ with:

$$\mathcal{T}_3^{(m)} = -\sum_{n=0}^{N-1} \delta t \sum_{K=[\overrightarrow{\sigma\sigma'}]\in\mathcal{M}} p_K^n\ (\varphi_{\sigma'}^{n+1} - \varphi_\sigma^{n+1}),$$
$$\mathcal{R}_3^{(m)} = -\sum_{n=0}^{N-1} \delta t \sum_{K=[\overrightarrow{\sigma\sigma'}]\in\mathcal{M}} (p_K^{n+1} - p_K^n)\ (\varphi_{\sigma'}^{n+1} - \varphi_\sigma^{n+1}).$$

The remainder term reads:

$$\mathcal{R}_3^{(m)} = \sum_{n=1}^{N-1} \delta t \sum_{K=[\overrightarrow{\sigma\sigma'}]\in\mathcal{M}} p_K^n\ \big[(\varphi_{\sigma'}^{n+1} - \varphi_\sigma^{n+1}) - (\varphi_{\sigma'}^n - \varphi_\sigma^n)\big] + \delta t \sum_{K=[\overrightarrow{\sigma\sigma'}]\in\mathcal{M}} p_K^0\ (\varphi_{\sigma'}^1 - \varphi_\sigma^1),$$

and thus:

$$|\mathcal{R}_3^{(m)}| \leq C_\varphi\ (\delta t^{(m)} + h^{(m)})\ \|p\|_{\mathrm{L}^\infty(\Omega\times(0,T))},$$

where the real number C_φ only depends on (the first and second derivatives of) φ. Thus $\mathcal{R}_3^{(m)}$ tends to zero when m tends to $+\infty$ and, since

$$\mathcal{T}_3^{(m)} = -\int_0^T \int_\Omega p^{(m)}\, \eth_x \varphi_{\mathcal{M}} \,\mathrm{d}x \,\mathrm{d}t,$$

we obtain that:

$$\lim_{m\to+\infty} T_3^{(m)} = \int_0^T \int_\Omega \bar p\, \partial_x \varphi \,\mathrm{d}x \,\mathrm{d}t.$$

Conclusion – Gathering the limits of all the terms of the mass and momentum balance equations concludes the proof. □

We now turn to the entropy condition (1.7). To this purpose, we need to introduce the following additional condition for a sequence of discretizations:

$$\lim_{m\to+\infty} \frac{\delta t^{(m)}}{\min_{K\in\mathcal{M}^{(m)}} h_K} = 0. \tag{5.15}$$

Note that this condition is slightly more restrictive that a standard CFL condition. It allows to bound the remainder term in the discrete elastic potential balance as stated in the following lemma.

Lemma 5.3. *Let Ω be an open bounded interval of $\mathbb{R}$. Let $(\mathcal{M}^{(m)},\ \delta t^{(m)})_{m\in\mathbb{N}}$ be a sequence of discretizations such that the time step $\delta t^{(m)}$ tends to zero as $m\to+\infty$, and let $(\rho^{(m)}, p^{(m)}, u^{(m)})_{m\in\mathbb{N}}$ be the corresponding sequence of solutions. We suppose that this sequence satisfies the estimates* (5.5) *and* (5.6). *In addition, we assume that $(\rho^{(m)})_{m\in\mathbb{N}}$ satisfies the following uniform BV estimate:*

$$\|\rho^{(m)}\|_{\mathcal{T},t,\mathrm{BV}} \leq C, \quad \forall m \in \mathbb{N}, \tag{5.16}$$

and, for $\gamma < 2$ only, is uniformly bounded by below, i.e. *that there exists $c > 0$ such that:*

$$c \leq (\rho^{(m)})_K^n, \quad \forall K \in \mathcal{M}^{(m)},\ \textit{for } 0 \leq n \leq N^{(m)},\ \forall m \in \mathbb{N}. \tag{5.17}$$

Let us suppose that the CFL condition (5.15) *holds. Let $\mathcal{R}^{(m)}$ be defined by:*

$$\mathcal{R}^{(m)} = \sum_{n=0}^{N-1} \delta t \sum_{K\in\mathcal{M}} (R_K^{n+1})^-,$$

with R_K^{n+1} given by (4.5)*. Then:*

$$\lim_{m \to +\infty} \mathcal{R}^{(m)} = 0.$$

Proof. For $K = [\overrightarrow{\sigma\sigma'}] \in \mathcal{M}$, with $\sigma = \overrightarrow{M|K}$ and $\sigma' = \overrightarrow{K|L}$, we write $R_K^{n+1} = (T_1)_K^{n+1} + (T_2)_K^{n+1} + (T_3)_K^{n+1}$, with:

$$\begin{aligned}
(T_1)_K^{n+1} &= \frac{1}{2} \frac{|K|}{\delta t}\ \mathcal{H}''(\overline{\rho}_{K,1}^n)\ (\rho_K^{n+1} - \rho_K^n)^2, \\
(T_2)_K^{n+1} &= \frac{1}{2} \Big[(u_{\sigma'}^n)^-\ \mathcal{H}''(\overline{\rho}_{\sigma'}^n)\ (\rho_K^n - \rho_L^n)^2 + (-u_\sigma^n)^-\ \mathcal{H}''(\overline{\rho}_\sigma^n)\ (\rho_K^n - \rho_M^n)^2 \Big], \\
(T_3)_K^{n+1} &= \Big[\rho_{\sigma'}^n\, u_{\sigma'}^n - \rho_\sigma^n\, u_\sigma^n \Big]\ \mathcal{H}''(\overline{\rho}_{K,2}^n)\ (\rho_K^{n+1} - \rho_K^n),
\end{aligned}$$

where $\overline{\rho}_{K,1}^n,\ \overline{\rho}_{K,2}^n \in [\![\rho_K^{n+1}, \rho_K^n]\!]$, $\overline{\rho}_{\sigma'}^n \in [\![\rho_K^n, \rho_L^n]\!]$ and $\overline{\rho}_\sigma^n \in [\![\rho_K^n, \rho_M^n]\!]$. The first two terms are non-negative, and thus $(R_K^{n+1})^- \leq |(T_3)_K^{n+1}|$. Since both ρ, u and, for $\gamma < 2$, $1/\rho$ are supposed to be bounded, we have:

$$\sum_{n=0}^{N-1} \delta t \sum_{K \in \mathcal{M}} |(T_3)_K^{n+1}| \leq C \frac{\delta t^{(m)}}{\min_{K \in \mathcal{M}} h_K}\ \|\rho^{(m)}\|_{\mathcal{T},t,\mathrm{BV}},$$

which yields the conclusion by the assumption (5.15). ◻

We are now in position to state the following consistency result.

Theorem 5.4 (Entropy consistency of the one dimensional scheme). *Let the assumptions of Theorem 5.2 hold. Let us suppose in addition that the considered sequence of discretizations satisfies* (5.15)*, and that $(\rho^{(m)})_{m \in \mathbb{N}}$ satisfies the BV estimate* (5.16) *and, for $\gamma < 2$, the uniform control* (5.17) *of $1/\rho^{(m)}$. Then the limit $(\bar\rho,\ \bar p,\ \bar u)$ satisfies the entropy condition* (1.7)*.*

Proof. Let $\varphi \in \mathrm{C}_c^\infty\big(\Omega \times [0, T)\big)$, $\varphi \geq 0$. With the notations for the interpolate of φ given in Definition 5.1, we multiply the kinetic balance equation (4.1)-(4.2) by φ_σ^{n+1}, and the elastic potential balance (4.4)-(4.5) by φ_K^{n+1}, sum over the edges and cells respectively and over the time steps, to obtain the discrete version of (1.7):

$$T_1^{(m)} + T_2^{(m)} + T_3^{(m)} + \tilde T_1^{(m)} + \tilde T_2^{(m)} + \tilde T_3^{(m)} = -R^{(m)} - \tilde R^{(m)} \tag{5.18}$$

where:

$$\begin{aligned}
T_1^{(m)} &= \sum_{n=0}^{N-1} \sum_{K \in \mathcal{M}} |K|\ \big[\mathcal{H}(\rho_K^{n+1}) - \mathcal{H}(\rho_K^n)\big]\ \varphi_K^{n+1}, \\
T_2^{(m)} &= \sum_{n=0}^{N-1} \delta t \sum_{K = [\overrightarrow{\sigma\sigma'}] \in \mathcal{M}} \big[\mathcal{H}(\rho_{\sigma'}^n)\, u_{\sigma'}^n - \mathcal{H}(\rho_\sigma^n)\, u_\sigma^n\big] \varphi_K^{n+1}, \\
T_3^{(m)} &= \sum_{n=0}^{N-1} \delta t \sum_{K = [\overrightarrow{\sigma\sigma'}] \in \mathcal{M}} \big[p_K^n (u_{\sigma'}^n - u_\sigma^n)\big]\ \varphi_K^{n+1}, \\
\tilde T_1^{(m)} &= \frac{1}{2} \sum_{n=0}^{N-1} \sum_{\sigma \in \mathcal{E}_{\mathrm{int}}} |D_\sigma|\ \big[\rho_{D_\sigma}^{n+1} (u_\sigma^{n+1})^2 - \rho_{D_\sigma}^n (u_\sigma^n)^2\big]\ \varphi_\sigma^{n+1}, \\
\tilde T_2^{(m)} &= \frac{1}{2} \sum_{n=0}^{N-1} \delta t \sum_{\sigma = \overrightarrow{K|L} \in \mathcal{E}_{\mathrm{int}}} \big[F_L^n\, (u_L^n)^2 - F_K^n\, (u_K^n)^2\big]\ \varphi_\sigma^{n+1}, \\
\tilde T_3^{(m)} &= \sum_{n=0}^{N-1} \delta t \sum_{\sigma = \overrightarrow{K|L} \in \mathcal{E}_{\mathrm{int}}} (p_L^{n+1} - p_K^{n+1})\, u_\sigma^{n+1}\ \varphi_\sigma^{n+1},
\end{aligned}$$

$$R^{(m)} = \sum_{n=0}^{N-1} \delta t \sum_{K\in\mathcal{M}} R_K^{n+1}\ \varphi_K^{n+1}, \qquad \tilde R^{(m)} = \sum_{n=0}^{N-1} \delta t \sum_{\sigma\in\mathcal{E}_{\rm int}} R_\sigma^{n+1}\ \varphi_\sigma^{n+1},$$

and the quantities R_K^{n+1} and R_σ^{n+1} are given by (the one-dimensional version of) Equation (4.5) and (4.2) respectively.

The fact that

$$\lim_{m\to+\infty} T_1^{(m)} = -\int_0^T \int_\Omega \mathcal{H}(\bar\rho)\ \partial_t \varphi \,\mathrm{d}x\,\mathrm{d}t - \int_\Omega \mathcal{H}(\rho_0)(x)\ \varphi(x,0)\,\mathrm{d}x,$$

is proven by the same technique as for passing to the limit in the term $T_1^{(m)}$ of the discrete mass balance equation in the proof Theorem 5.2, thanks to the fact that, with the assumed convergence of the sequence $(\rho^{(m)})_{m\in\mathbb{N}}$, the sequence $(\mathcal{H}(\rho^{(m)}))_{m\in\mathbb{N}}$ converge to $\mathcal{H}(\bar\rho)$ in $\mathrm{L}^r(\Omega\times(0,T))$, for $r\geq 1$. For $T_2^{(m)}$, we have, reordering the sums:

$$T_2^{(m)} = -\sum_{n=0}^{N-1} \delta t \sum_{\sigma=\overrightarrow{K|L}\in\mathcal{E}_{\rm int}} \mathcal{H}(\rho_\sigma^n)\, u_\sigma^n\ (\varphi_L^{n+1} - \varphi_K^{n+1}).$$

Let us write $T_2^{(m)} = \mathcal{T}_2^{(m)} + \mathcal{R}_2^{(m)}$, with

$$\mathcal{T}_2^{(m)} = -\sum_{n=0}^{N-1} \delta t \sum_{\sigma=\overrightarrow{K|L}\in\mathcal{E}_{\rm int}} \big(|D_{K,\sigma}|\ \mathcal{H}(\rho_K^n) + |D_{L,\sigma}|\ \mathcal{H}(\rho_L^n)\big)\, u_\sigma^n\ \frac{\varphi_L^{n+1} - \varphi_K^{n+1}}{h_\sigma}.$$

We have:

$$\mathcal{T}_2^{(m)} = -\int_0^T \int_\Omega \mathcal{H}(\rho^{(m)})\ u^{(m)}\ \eth_x \varphi_{\mathcal{M}} \,\mathrm{d}x\,\mathrm{d}t,$$

so

$$\lim_{m\to+\infty} T_2^{(m)} = -\int_0^T \int_\Omega \mathcal{H}(\bar\rho)\ \bar u\ \partial_x \varphi \,\mathrm{d}x\,\mathrm{d}t.$$

The remainder term $\mathcal{R}_2^{(m)}$ reads:

$$\mathcal{R}_2^{(m)} = -\sum_{n=0}^{N-1} \delta t \sum_{\sigma=\overrightarrow{K|L}\in\mathcal{E}_{\rm int}} \big[|D_\sigma|\ \mathcal{H}(\rho_\sigma^n) - |D_{K,\sigma}|\ \mathcal{H}(\rho_K^n) - |D_{L,\sigma}|\ \mathcal{H}(\rho_L^n)\big]\, u_\sigma^n\ \frac{\varphi_L^{n+1} - \varphi_K^{n+1}}{h_\sigma}.$$

This term satisfies:

$$|\mathcal{R}_2^{(m)}| \leq \sum_{n=0}^{N-1} \delta t \sum_{\sigma=\overrightarrow{K|L}\in\mathcal{E}_{\rm int}} |\mathcal{H}(\rho_K^n) - \mathcal{H}(\rho_L^n)|\ u_\sigma^n\ |\varphi_L^{n+1} - \varphi_K^{n+1}|,$$

and so

$$|\mathcal{R}_2^{(m)}| \leq C_\varphi\ h^{(m)}\ \|u^{(m)}\|_{\mathrm{L}^\infty(\Omega\times(0,T))}\ \|\rho^{(m)}\|_{\mathcal{T},x,\mathrm{BV}},$$

provided that a uniform (with respect to the faces, the time steps and the meshes) Lipschitz condition holds for $|\mathcal{H}(\rho_K^n) - \mathcal{H}(\rho_L^n)|$ which, in view of the expression of $\mathcal{H}$, requires that the sequence $(\rho^{(m)})_{m\in\mathbb{N}}$ be bounded by below away from zero when $\gamma=1$.

For the other terms at the left-hand side of (5.18), we refer to [15, Theorem 5.3]. Finally, the remainder term $R^{(m)}$ is non-negative under the CFL condition (4.3), while

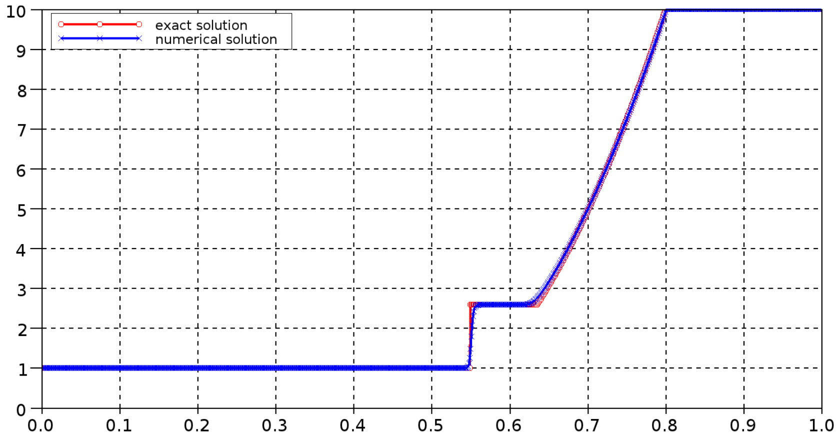

FIG. 6.1. *Test 1 – $h = 0.001$, $\delta t = h/12$ – Density at $t = 0.025$.*

the positive part of $\tilde{R}^{(m)}$ tends to zero in $\mathrm{L}^1(\Omega \times (0,T))$ under the assumption (5.15) by Lemma 5.3. The proof is thus complete. □

REMARK 5.1 (On BV-stability assumptions).
The proof of Theorem 5.2 shows that the scheme is consistent under a BV-stability assumption much weaker than (5.7)*, namely:*

$$\lim_{m\to+\infty} h^{(m)} \left[\|\rho^{(m)}\|_{\mathcal{T},x,\mathrm{BV}} + \|u^{(m)}\|_{\mathcal{T},x,\mathrm{BV}}\right] = 0.$$

*The situation is completely different when prooving that the limit of convergent sequences is an entropy solution (*i.e.* when prooving Theorem 5.4 or, more precisely speaking, the preliminary lemma 5.3), since we need:*

$$\lim_{m\to+\infty} \frac{\delta t^{(m)}}{\min_{K\in\mathcal{M}^{(m)}} h_K} \|\rho^{(m)}\|_{\mathcal{T},t,\mathrm{BV}} = 0.$$

6. Numerical results. We assess in this section the behaviour of the scheme on various test cases. For all these tests, we chose $p = \rho^2$ for the equation of state, so the solved system turns out to be the so-called shallow water equations.

6.1. A first Riemann problem. We begin with a Riemann problem, *i.e.* a 1D problem which initial conditions consists in two constant states separated by a discontinuity. The chosen left and right states are given by:

$$\text{left state: } \begin{bmatrix} \rho_L = 1 \\ u_L = 5 \end{bmatrix}; \qquad \text{right state: } \begin{bmatrix} \rho_R = 10 \\ u_R = 7.5 \end{bmatrix}.$$

The computational domain is $\Omega = (0,1)$ and the final time is $T = 0.025$. The (known) analytical solution of this problem consists, from the left to the right, in a shock wave and a rarefaction wave, both travelling to the right, separated by constant states.

6.1.1. Results. The density and velocity obtained at $t = 0.025 = T$ are shown of Figures 6.1 and 6.2 respectively; this results have been obtain with $h = 0.001$ and $\delta t = h/12$ (the maximum velocity and sound speed computed from the analytical solution being $u_{\max} = 7.5$ and $c_{\max} \simeq 4.5$, respectively). In addition, we performed a convergence study, successively dividing by two the space and time steps (so keeping the CFL number constant). The difference between the computed and analytical solution at $t = 0.025$, measured in $\mathrm{L}^1(\Omega)$ norm, are reported in the following table:

space step	$h_0 = 1/250$	$h_0/2$	$h_0/4$	$h_0/8$	$h_0/16$
$\|\rho - \bar{\rho}\|_{\mathrm{L}^1(\Omega)}$	0.0449	0.0256	0.0135	0.00775	0.00429
$\|u - \bar{u}\|_{\mathrm{L}^1(\Omega)}$	0.0411	0.0233	0.0119	0.00696	0.00384

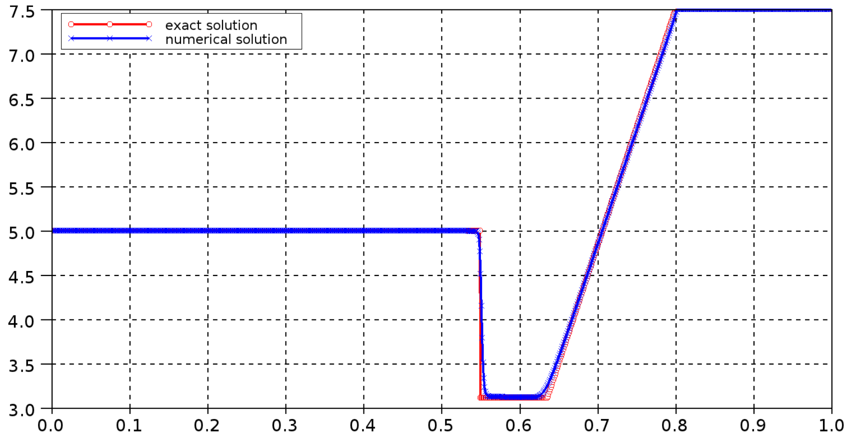

FIG. 6.2. *Test 1 – $h = 0.001$, $\delta t = h/12$ –Velocity at $t = 0.025$.*

We observe an approximatively first-order convergence rate.

To complete the study, we performed a computation of the same problem, but subtracting a constant real number to the left and right velocity, in such a way that the velocity on the intermediate state nearly vanishes. In this case, we observe spurious oscillations on the solution, probably due to the fact that the numerical diffusion in the scheme vanishes. However, adding an artificial viscosity term in the discrete momentum balance equation, with a constant viscosity equal to $0.5\,\rho\,h$ (so equal to the upwind viscosity which would be associated to a velocity equal to 1) completely cures the problem. This observation strongly supports the idea to build a higher order scheme using an *a posteriori* fitted viscosity technique, as in the so-called entropy viscosity method [8, 9]; this work is underway.

6.1.2. On a naive scheme. We also test the "naive" explicit scheme obtained by evaluating all the terms, except of course the time-derivative one, at time t^n. In the one dimensional setting and with the same notations as in Section 5, this scheme thus reads:

$$\forall K = [\overrightarrow{\sigma\sigma'}] \in \mathcal{M}, \qquad \frac{|K|}{\delta t}(\rho_K^{n+1} - \rho_K^n) + F_{\sigma'}^n - F_\sigma^n = 0, \tag{6.1a}$$

$$\forall \sigma = \overrightarrow{K|L} \in \mathcal{E}_{\rm int}, \qquad \frac{|D_\sigma|}{\delta t}(\rho_{D_\sigma}^{n+1} u_\sigma^{n+1} - \rho_{D_\sigma}^n u_\sigma^n) + F_L^n u_L^n - F_K^n u_K^n + p_L^n - p_K^n = 0, \tag{6.1b}$$

$$\forall K \in \mathcal{M}, \qquad p_K^{n+1} = \wp(\rho_K^{n+1}) = (\rho_K^{n+1})^\gamma. \tag{6.1c}$$

Hereafter and on the figure captions, this scheme is referred to as the "$\rho \rightsquigarrow u \rightsquigarrow p$ scheme" (since the pressure is updated after the computation of the velocity rather than after the computation of the density).

The computed density and velocity at time $T = 0.025$ are plotted on figures 6.3 and 6.4 respectively. From these results, it appears clearly that the $\rho \rightsquigarrow u \rightsquigarrow p$ scheme generates discontinuities in the rarefaction wave, and further experiments show that this phenomenon is not cured by a decrease of the time and space steps; this seems to be connected to the fact that, for this variant, we cannot prove that the limits of converging sequences satisfy the entropy condition (in fact, they probably do not). When trying to do so, in our proof and from a purely technical point of view, the trouble comes from the fact that the pressure gradient term which appears in the kinetic energy balance reads $\boldsymbol{u}^{n+1}\boldsymbol{\nabla} p^n$ and it seems difficult to make the counterpart (*i.e.* $p^n {\rm div}(\boldsymbol{u}^{n+1})$) appear, with the corresponding time levels, in the elastic potential balance, starting from a mass balance with a convection term written with $\boldsymbol{u}^n$; hence

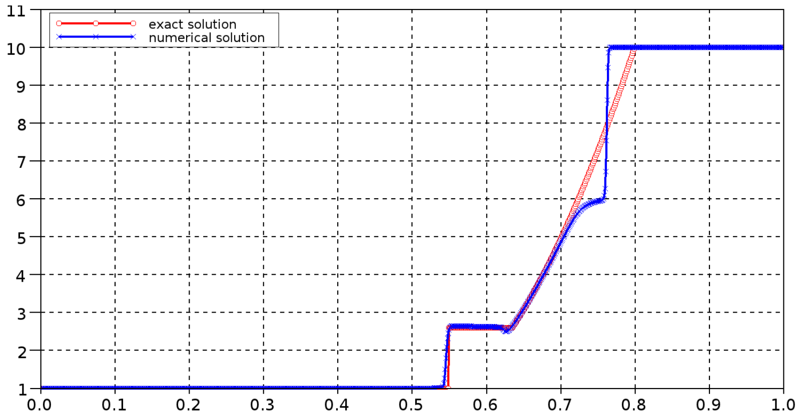

FIG. 6.3. *Test 1, $\rho \rightsquigarrow u \rightsquigarrow p$ scheme – $h = 0.001$, $\delta t = h/12$ – Density at $t = 0.025$.*

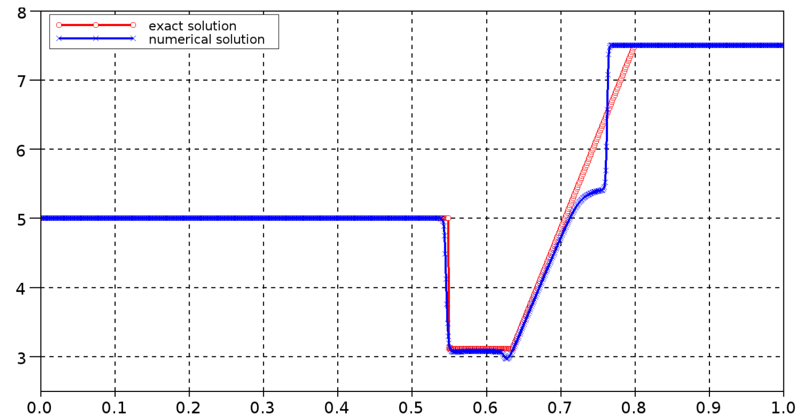

FIG. 6.4. *Test 1, $\rho \rightsquigarrow u \rightsquigarrow p$ scheme – $h = 0.001$, $\delta t = h/12$ – Velocity at $t = 0.025$.*

a dicretization of the momentum balance equation with an updated pressure gradient term $\boldsymbol{\nabla} p^{n+1}$, and thus the inversion of steps in the algorithm, to get the actual scheme proposed in this paper.

6.2. Problems involving vacuum zones in the flow. The objective of the two tests presented in this section is to check that the time step does not have to be drastically reduced in the presence of vacuum. Both are Riemann problems, posed on $\Omega = (0, 1)$.

We first begin with a case where the vacuum is initially present, at the right initial state:

$$\text{left state: } \begin{bmatrix} \rho_L = 1 \\ u_L = 1 \end{bmatrix}; \qquad \text{right state: } \begin{bmatrix} \rho_R = 0 \\ u_R = 0 \end{bmatrix}.$$

In the computer code, ρ_R is fixed as $\rho_R = 10^{-20}$, to prevent divisions by zero due to imprudent programming. The results obtained at $t = 0.05$ are plotted on Figure 6.5 (density) and Figure 6.6 (velocity); they have been obtained with $h = 0.001$ and a constant time step equal to $\delta t = h/8$, which seems to be near to the stability limit (the maximum velocity and sound speed computed from the analytical solution being given by $u_{\max} \simeq 3.8$ and $c_{\max} \simeq 1.4$, respectively). We observe that the prediction velocity is rather poor near to the vacuum front; we however check on Figure 6.7 that the scheme converges to the right solution; moreover, Figure 6.8 shows that the quantity $\rho\, u$ (which is, in this case, the quantity of physical interest) is in fact obtained with a reasonable accuracy with the coarsest meshes of this study.

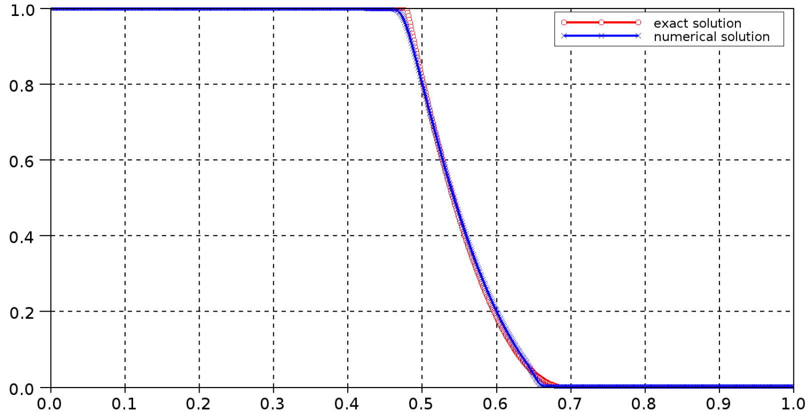

FIG. 6.5. *Riemann problem with vacuum at the right state – $h = 0.001$, $\delta t = h/8$ – Density at* $t = 0.05$.

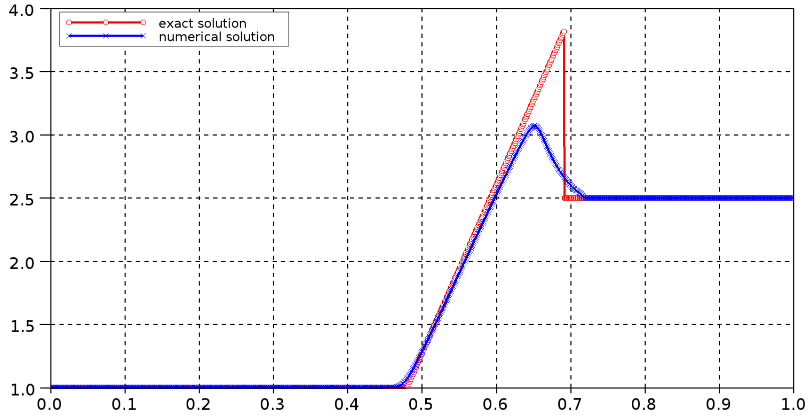

FIG. 6.6. *Riemann problem with vacuum at the right state – $h = 0.001$, $\delta t = h/8$ – Velocity at* $t = 0.05$.

We now turn to a case where the chosen left and right states are given by:

$$\text{left state: } \begin{bmatrix} \rho_L = 1 \\ u_L = -8 \end{bmatrix}; \qquad \text{right state: } \begin{bmatrix} \rho_R = 1 \\ u_R = 8 \end{bmatrix}.$$

In this case, the solution consists in an intermediate state corresponding to vacuum connected to the left and right initial states by rarefaction waves. The computed density and velocity at $t = 0.03$, with $h = 0.001$ and $\delta t = h/12$ (while, in the analytical solution, $u_{\max} = 8$ and $c_{\max} \simeq 1.4$), are plotted on Figures 6.9 and 6.10 respectively. Once again, the behaviour of the scheme is satisfactory.

7. Conclusion. We presented in this paper an explicit scheme based on staggered meshes for the hyperbolic system of the barotropic Euler equations. This algorithm uses a very simple first-order upwinding strategy which consists, equation by equation, to implement an upwind discretization of the convection term with respect of the material velocity. Under CFL-like conditions based on the material velocity only (by opposition to the celerity of waves), this scheme preserves the positivity of the density and the pressure, and has been shown to be consistent for 1D problems, in the sense that, if a sequence of numerical solutions obtained with more and more refined meshes (and, accordingly, smaller and smaller time steps) converges, then the limit is a weak entropy solution to the continuous problem. This theoretical result may probably be extended to the multi-dimensional case, and this work is now being

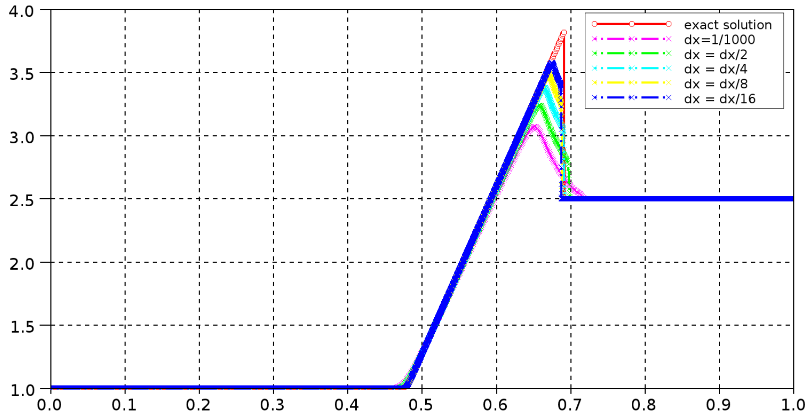

FIG. 6.7. *Riemann problem with vacuum at the right state – $h = h_0 = 0.001$ to $h = h_0/16$, $\delta t = h/8$ – Velocity at $t = 0.05$.*

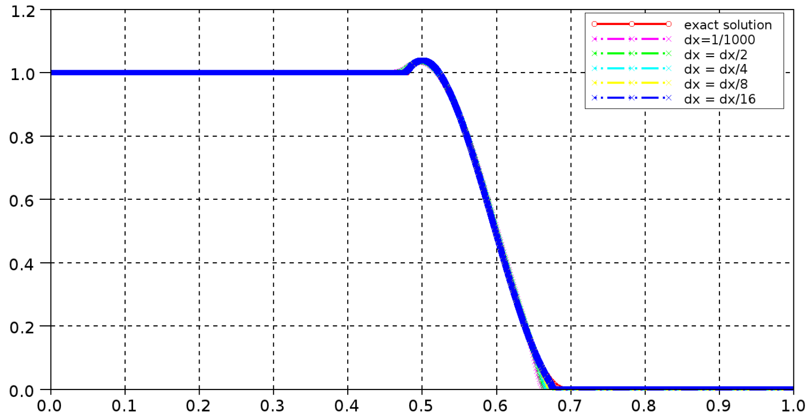

FIG. 6.8. *Riemann problem with vacuum at the right state – $h = h_0 = 0.001$ to $h = h_0/16$, $\delta t = h/8$ – Mass flowrate at $t = 0.05$.*

undertaken. The proposed scheme has a natural extension to the full Euler equations, which is the topic of a companion paper. Note also that a partial time-implicitation, using pressure correction techniques, has been shown to yield consistent unconditionally stable schemes [12, 13].

Numerical studies show that the proposed algorithm is stable, even if the largest time step before blow-up is smaller than suggested by the above-mentionned CFL conditions. This behaviour had to be expected, since these CFL conditions only involve the velocity (and not the celerity of the acoutic waves): indeed, were they the only limitation, we would have obtained an explicit scheme stable up to the incompressible limit. However, the mechanisms leading to the blow-up of the scheme (or, conversely, the way to fix the time step to ensure stability) remain to be understood.

In addition, numerical experiments show that some oscillations appear near stagnation points, where the numerical diffusion brought by the upwinding vanishes. These oscillations are damped by a small amount of artificial (physical-like) viscosity, and this suggests to implement techniques consisting in adding to the scheme such a diffusion term, with a viscosity monitored by an *a posteriori* (*i.e.* performed in view of the results of the previous time step) analysis of the solution, as the so-called entropy-viscosity technique. Besides, such an extension should allow to design a more accurate scheme, based on higher-order numerical fluxes. This work is underway.

Last but not least, since the proposed scheme uses very simple numerical fluxes,

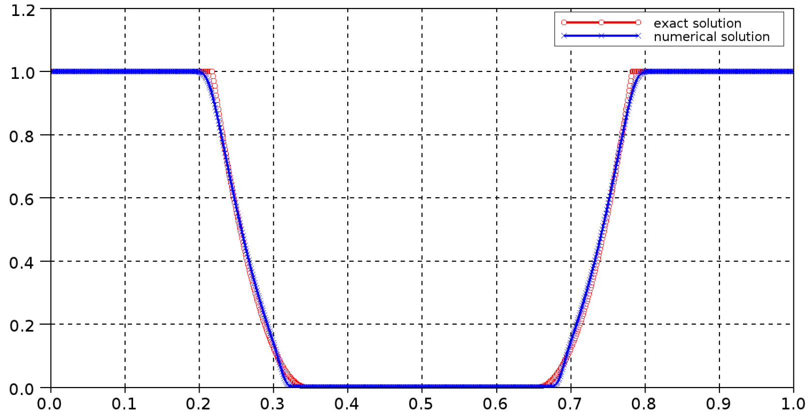

FIG. 6.9. *Riemann problem with vacuum appearance – $h = 0.001$, $\delta t = h/12$ – Density at $t = 0.03$.*

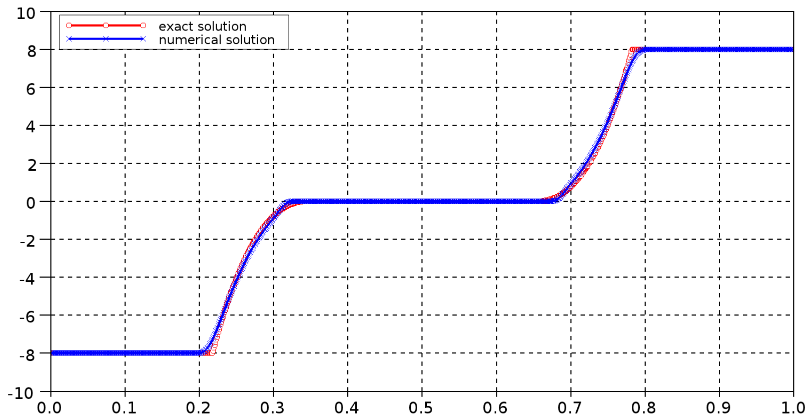

FIG. 6.10. *Riemann problem with vacuum appearance – $h = 0.001$, $\delta t = h/12$ – Mass flowrate at $t = 0.03$.*

it is well suited to large multi-dimensionnal parallel computing applications. This is the topic of ongoing studies at IRSN.

Appendix A. Some results concerning explicit finite volume convection operators.

We begin with the convection operator appearing in the mass balance equation, which reads, in the continuous problem, $\rho \to \mathcal{C}(\rho) = \partial_t \rho + \mathrm{div}(\rho \boldsymbol{u})$, where $\boldsymbol{u}$ stands for a given velocity field, which is not assumed to satisfy any divergence constraint. Let ψ be a regular function from $(0, +\infty)$ to $\mathbb{R}$; then:

$$\begin{aligned} \psi'(\rho)\, \mathcal{C}(\rho) = \psi'(\rho)\, \partial_t(\rho) + \psi'(\rho)\, \boldsymbol{u} \cdot \boldsymbol{\nabla} \rho + \psi'(\rho)\, \rho\, \mathrm{div} \boldsymbol{u} \\ = \partial_t(\psi(\rho)) + \boldsymbol{u} \cdot \boldsymbol{\nabla} \psi(\rho) + \rho\, \psi'(\rho)\, \mathrm{div} \boldsymbol{u}, \end{aligned}$$

so adding and subtracting $\psi(\rho)\,\mathrm{div}\boldsymbol{u}$ yields:

$$\psi'(\rho)\ \mathcal{C}(\rho) = \partial_t\big(\psi(\rho)\big) + \mathrm{div}\big(\psi(\rho)\boldsymbol{u}\big) + \big(\rho\psi'(\rho) - \psi(\rho)\big)\ \mathrm{div}\boldsymbol{u}. \qquad \text{(A.1)}$$

This computation is of course completely formal and only valid for regular functions ρ and $\boldsymbol{u}$. The following lemma states a discrete analogue to (A.1).

LEMMA A.1. *Let P be a polygonal (resp. polyhedral) bounded set of $\mathbb{R}^2$ (resp. $\mathbb{R}^3$), and let $\mathcal{E}(P)$ be the set of its edges (resp. faces). Let ψ be a twice continuously differentiable function defined over $(0, +\infty)$. Let $\rho_P^* > 0$, $\rho_P > 0$, $\delta t > 0$; consider* *three families $(\rho^*_\eta)_{\eta\in\mathcal{E}(P)} \subset \mathbb{R}_+ \setminus \{0\}$, $(V^*_\eta)_{\eta\in\mathcal{E}(P)} \subset \mathbb{R}$ and $(F^*_\eta)_{\eta\in\mathcal{E}(P)} \subset \mathbb{R}$ such that*

$$\forall \eta \in \mathcal{E}(P), \qquad F^*_\eta = \rho^*_\eta \; V^*_\eta.$$

Let $R_{P,\delta t}$ be defined by:

$$\begin{aligned} R_{P,\delta t} = \Big[&\frac{|P|}{\delta t}\; (\rho_P - \rho^*_P) + \sum_{\eta\in\mathcal{E}(P)} F^*_\eta \Big] \; \psi'(\rho_P) \\ &- \frac{|P|}{\delta t}\; [\psi(\rho_P) - \psi(\rho^*_P)] + \sum_{\eta\in\mathcal{E}(P)} \psi(\rho^*_\eta) V^*_\eta + [\rho^*_P \psi'(\rho^*_P) - \psi(\rho^*_P)] \sum_{\eta\in\mathcal{E}(P)} V^*_\eta. \end{aligned}$$

Then this quantity may be expressed as follows:

$$\begin{aligned} R_{P,\delta t} = \frac{1}{2}\frac{|P|}{\delta t}\,(\rho_P - \rho^*_P)^2\, \psi''(\overline{\rho}^{(1)}_P) - \frac{1}{2} \sum_{\eta\in\mathcal{E}(P)} V^*_\eta\, (\rho^*_P - \rho^*_\eta)^2\, \psi''(\overline{\rho}^*_\eta) \\ + \sum_{\eta\in\mathcal{E}(P)} V^*_\eta \rho^*_\eta\, (\rho_P - \rho^*_P)\, \psi''(\overline{\rho}^{(2)}_P), \end{aligned}$$

where $\overline{\rho}^{(1)}_P$, $\overline{\rho}^{(2)}_P \in [\![\rho_P, \rho^*_P]\!]$ and $\forall \eta \in \mathcal{E}(P)$, $\overline{\rho}^*_\eta \in [\![\rho^*_P, \rho^*_\eta]\!]$. We recall that, for $a,\ b \in \mathbb{R}$, we denote by $[\![a,b]\!]$ the interval $[\![a,b]\!] = \{\theta a + (1-\theta) b,\ \theta \in [0,1]\}$.

Proof. By the definition of F^*_η, we have:

$$\begin{aligned} \Big[\frac{|P|}{\delta t}(\rho_P - \rho^*_P) + \sum_{\eta\in\mathcal{E}(P)} F^*_\eta\Big]\; \psi'(\rho_P) = \frac{|P|}{\delta t}(\rho_P - \rho^*_P)\, \psi'(\rho_P) \\ + \sum_{\eta\in\mathcal{E}(P)} \rho^*_\eta V^*_\eta \psi'(\rho^*_P) + \sum_{\eta\in\mathcal{E}(P)} \rho^*_\eta V^*_\eta \big[\psi'(\rho_P) - \psi'(\rho^*_P)\big]. \end{aligned} \tag{A.2}$$

By Taylor expansions of ψ, there exist two real numbers $\overline{\rho}^{(1)}_P$ and $\overline{\rho}^{(2)}_P \in [\![\rho^*_P, \rho_P]\!]$ and a family of real numbers $(\overline{\rho}^*_\eta)_{\eta\in\mathcal{E}(P)}$ satisfying, $\forall \eta \in \mathcal{E}(P)$, $\overline{\rho}^*_\eta \in [\![\rho^*_P, \rho^*_\eta]\!]$, and such that:

$$\begin{aligned} &(\rho_P - \rho^*_P)\, \psi'(\rho_P) = \psi(\rho_P) - \psi(\rho^*_P) + \frac{1}{2}\,(\rho_P - \rho^*_P)^2\, \psi''(\overline{\rho}^{(1)}_P), \\ &\rho^*_\eta \psi'(\rho^*_P) = \psi(\rho^*_\eta) + [\rho^*_P \psi'(\rho^*_P) - \psi(\rho^*_P)] - \frac{1}{2}\,(\rho^*_\eta - \rho^*_P)^2\, \psi''(\overline{\rho}^*_\eta), \\ &\psi'(\rho_P) - \psi'(\rho^*_P) = (\rho_P - \rho^*_P)\, \psi''(\overline{\rho}^{(2)}_P). \end{aligned}$$

Substituting in (A.2) yields the result we are seeking. ▯

We now turn to the convection operator appearing in the momentum balance equation, which reads, in the continuous setting, $z \to \mathcal{C}_\rho(z) = \partial_t(\rho z) + \mathrm{div}(\rho z \boldsymbol{u})$, where ρ (resp. $\boldsymbol{u}$) stands for a given scalar (resp. vector) field; we wish to obtain some property of C_ρ under the assumption that ρ and $\boldsymbol{u}$ satisfy the mass balance equation, *i.e.* $\partial_t \rho + \mathrm{div}(\rho \boldsymbol{u}) = 0$. Formally, using twice the mass balance yields:

$$\begin{aligned} \psi'(z)\; \mathcal{C}_\rho(z) = \psi'(z)\big[\partial_t(\rho\, z) + \mathrm{div}(\rho\, z\, \boldsymbol{u})\big] = \psi'(z)\rho\big[\partial_t z + \boldsymbol{u}\cdot \boldsymbol{\nabla} z\big] \\ = \rho\big[\partial_t \psi(z) + \boldsymbol{u}\cdot\boldsymbol{\nabla}\psi(z)\big] = \partial_t\big(\rho\, \psi(z)\big) + \mathrm{div}\big(\rho\, \psi(z)\, \boldsymbol{u}\big). \end{aligned}$$

Taking for z a component of the velocity field, this relation is the central argument used to derive the kinetic energy balance. The following lemma states a discrete counterpart of this identity, for a finite volume first-order explicit convection operator.

LEMMA A.2. *Let P be a polygonal (resp. polyhedral) bounded set of $\mathbb{R}^2$ (resp. $\mathbb{R}^3$) and let $\mathcal{E}(P)$ be the set of its edges (resp. faces). Let $\rho^*_P > 0$, $\rho_P > 0$, $\delta t > 0$, and $(F^*_\eta)_{\eta\in\mathcal{E}(P)} \subset \mathbb{R}$ be such that*

$$\frac{|P|}{\delta t}\; (\rho_P - \rho^*_P) + \sum_{\eta\in\mathcal{E}(P)} F^*_\eta = 0. \tag{A.3}$$

Let ψ be a twice continuously differentiable function defined over $(0,+\infty)$. For $u_P^* \in \mathbb{R}$, $u_P \in \mathbb{R}$ and $(u_\eta^*)_{\eta\in\mathcal{E}(P)} \subset \mathbb{R}$ let us define:

$$R_{P,\delta t} = \Big[\frac{|P|}{\delta t}\,\big(\rho_P\,u_P - \rho_P^*\,u_P^*\big) + \sum_{\eta\in\mathcal{E}(P)} F_\eta^*\;u_\eta^*\Big]\;\psi'(u_P)$$
$$-\Big[\frac{|P|}{\delta t}\;\big[\rho_P\,\psi(u_P) - \rho_P^*\,\psi(u_P^*)\big] + \sum_{\eta\in\mathcal{E}(P)} F_\eta^*\;\psi(u_\eta^*)\Big].$$

Then:

(i) the remainder term $R_{P,\delta t}$ reads:

$$R_{P,\delta t} = \frac{1}{2}\,\frac{|P|}{\delta t}\,\rho_P\,(u_P - u_P^*)^2\psi''(\overline{u}_P^{(1)}) - \frac{1}{2}\sum_{\eta\in\mathcal{E}(P)} F_\eta^*\,(u_\eta^* - u_p^*)^2\psi''(\overline{u}_\eta^*)$$
$$+\sum_{\eta\in\mathcal{E}(P)} F_\eta^*\,(u_\eta^* - u_P^*)\;(u_P - u_P^*)\;\psi''(\overline{u}_P^{(2)}) \tag{A.4}$$

with $\overline{u}_P^{(1)},\overline{u}_P^{(2)} \in [\![u_P, u_P^*]\!]$, and $\forall\eta\in\mathcal{E}(P)$, $\overline{u}_\eta^* \in [\![u_P^*, u_\eta^*]\!]$.

(ii) If we suppose that the function ψ is convex and that $u_\eta^* = u_P^*$ as soon as $F_\eta^* \geq 0$, then $R_{P,\delta t}$ is non-negative under the CFL condition:

$$\delta t \leq \frac{|P|\,\rho_P\,\underline{\psi}''_P}{\sum_{\eta\in\mathcal{E}(P)}(F_\eta^*)^-\;(\overline{\psi}''_P)^2/\underline{\psi}''_\eta}, \tag{A.5}$$

where $\underline{\psi}''_P = \min_{s\in[\![u_P,u_P^*]\!]}\psi''(s)$, $\overline{\psi}''_P = \max_{s\in[\![u_P,u_P^*]\!]}\psi''(s)$ and $\underline{\psi}''_\eta = \min_{s\in[\![u_P^*,u_\eta^*]\!]}\psi''(s)$. For $\psi(s) = s^2/2$ (and therefore $\psi''(s)=1$, $\forall s\in(0,+\infty)$), this CFL condition simply reads:

$$\delta t \leq \frac{|P|\,\rho_P}{\sum_{\eta\in\mathcal{E}(P)}(F_\eta^*)^-}. \tag{A.6}$$

Proof. Let T_P be defined by:

$$T_P = \Big[\frac{|P|}{\delta t}\;\big(\rho_P\,u_P - \rho_P^*\,u_P^*\big) + \sum_{\eta\in\mathcal{E}(P)} F_\eta^*\;u_\eta^*\Big]\;\psi'(u_P).$$

Using equation (A.3) multiplied by u_P^*, we obtain:

$$T_P = \Big[\frac{|P|}{\delta t}\;\rho_P\;\big(u_P - u_P^*\big) + \sum_{\eta\in\mathcal{E}(P)} F_\eta^*\;(u_\eta^* - u_P^*)\Big]\;\psi'(u_P).$$

We now define the remainder terms r_P and $(r_\eta^*)_{\eta\in\mathcal{E}(P)}$ by:

$$r_P = (u_P - u_P^*)\;\psi'(u_P) - \big[\psi(u_P) - \psi(u_P^*)\big],\quad r_\eta^* = (u_P^* - u_\eta^*)\;\psi'(u_P^*) - \big[\psi(u_P^*) - \psi(u_\eta^*)\big].$$

With these notations, we get:

$$T_P = \frac{|P|}{\delta t}\;\rho_P\;\big[\psi(u_P) - \psi(u_P^*)\big] + \sum_{\eta\in\mathcal{E}(P)} F_\eta^*\big[\psi(u_\eta^*) - \psi(u_P^*)\big]$$
$$+\frac{|P|}{\delta t}\;\rho_P\;r_P - \sum_{\eta\in\mathcal{E}(P)} F_\eta^*\;r_\eta^* + \sum_{\eta\in\mathcal{E}(P)} F_\eta^*\;(u_\eta^* - u_P^*)\;\big(\psi'(u_P) - \psi'(u_P^*)\big).$$

Using once again equation (A.3), this time multiplied by $\psi(u_P^*)$, we obtain:

$$T_P = \frac{|P|}{\delta t} \left[\rho_P \psi(u_P) - \rho_P^* \psi(u_P^*)\right] + \sum_{\eta \in \mathcal{E}(P)} F_\eta^* \psi(u_\eta^*)$$
$$+ \frac{|P|}{\delta t}\, \rho_P\; r_P - \sum_{\eta \in \mathcal{E}(P)} F_\eta^*\; r_\eta^* + \sum_{\eta \in \mathcal{E}(P)} F_\eta^*\; (u_\eta^* - u_P^*)\; \big(\psi'(u_P) - \psi'(u_P^*)\big).$$

The expression (A.4) of the remainder term $R_{P,\delta t}$ follow by remarking that, by a Taylor expansion, there exist $\overline{u}_P^{(1)}, \overline{u}_P^{(2)} \in [\![u_P, u_P^*]\!]$, and $\forall \eta \in \mathcal{E}(P)$, $\overline{u}_\eta^* \in [\![u_P^*, u_\eta^*]\!]$ such that:

$$r_P = \frac{1}{2}\; \psi''(\overline{u}_P^{(1)})\,(u_P - u_P^*)^2, \quad r_\eta^* = \frac{1}{2}\; \psi''(\overline{u}_\eta^*)\,(u_\eta^* - u_p^*)^2$$

and

$$\psi'(u_P) - \psi'(u_P^*) = \psi''(\overline{u}_P^{(2)})\,(u_P - u_P^*).$$

If ψ is convex, r_P is non-negative. If, in addition, $u_P^* - u_\eta^*$ vanishes $\forall \eta \in \mathcal{E}(P)$ when F_η^* is non-negative, $-r_\eta^*$ is non-negative. By Young's inequality, the last term in $R_{P,\delta t}$ may be bounded as follows:

$$\Big|\sum_{\eta \in \mathcal{E}(P)} (F_\eta^*)^-\; (u_\eta^* - u_P^*)\; (u_P - u_P^*)\; \psi''(\overline{u}_P^{(2)})\Big|$$
$$\leq \frac{\psi''(\overline{u}_P^{(2)})^2}{2}\Big[\sum_{\eta \in \mathcal{E}(P)} (F_\eta^*)^-\; \frac{1}{\psi''(\overline{u}_\eta^*)}\Big]\; (u_P - u_P^*)^2 + \frac{1}{2} \sum_{\eta \in \mathcal{E}(P)} (F_\eta^*)^-\; (u_\eta^* - u_P^*)^2\; \psi''(\overline{u}_\eta^*),$$

so this term may be absorbed in the first two ones under the CFL condition (A.5). □

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